Sci. Annals, Fac. Sciences, Univ. Thessaloniki, **23**, 21 (1983)

A STUDY OF THE 16 DIMENSIONAL BHABHA WAVE EQUATION AND ITS CHARGE

By

C. G. KOUTROULOS*

Department of Theoretical Physics
University of St. Andrews
St. Andrews Fife, SCOTLAND

Abstract: *A detailed expositions is given of calculations reported previously in ref. (1) and in addition it is verified by the method adopted namely the Lie algebraic one that the charge associated with the 16 dimensionsl Bhabha field is indefinite.*

1. INTRODUCTION

In a previous paper[1] (refered to as (I)) it was shown that the 20 dimensional matrices L_{o}^{20-dim}, (k = 0,1,2,3) appearing in the Bhabha wave equation[2] based on the 20 dimensional representation of the group SO(4,1) appropriate for the description of spin 3/2 particles are given by the linear combinations (27), (28), (29) and (30) of ret. (I). In the present paper we shall study the 16 dimensional representation ot the group SO(4,1) in connection with the spin 3/2 Bhabha field. In this case the above linear combinations read:

---

* Present address: Department of Theoretical Physics University of Thessaloniki Thessaloniki, GREECE

$$L_0^{16-\dim} = i\sqrt{3}\,\Gamma_{16-\dim}\,(\vec{e}_{a_2}) + i\sqrt{3}\,\Gamma_{16-\dim}(\vec{e}_{-a_2})$$

$$L_1^{16-\dim} = i\sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}\,(\vec{e}_{a_1}) - i\sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}(\vec{e}_{-a_1}) +$$

$$+ i\sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}(\vec{e}_{a_1+2a_2}) - i\sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}(\vec{e}_{-a_1-2a_2})\,,$$

$$L_2^{16-\dim} = -\sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}\,(\vec{e}_{a_1}) - \sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}(\vec{e}_{-a_1}) -$$

$$-\sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}(\vec{e}_{a_1+2a_2}) - \sqrt{\frac{3}{2}}\,\Gamma_{16-\dim}(\vec{e}_{-a_1-2a_2})\,,$$

$$L_3^{16-\dim} = -6i\,\Gamma_{16-\dim}(\vec{h}_{a_2}).$$

We shall see also how the hermitianizing matrix transforms under the transformations of the group SO(4,1) which also belong to the group SO(3,1) and finally using the eigenvalues of $L_0^{16-\dim}$ and the hermitianizing matrix $A_{16\ \dim}$ we shall calculate the charge density associate with the 16 - dimensional Bhabha field.

2. 16 - DIMENSIONAL REPRESENTATION OF B_2

Using the dimensionality formula:

$$N = \frac{1}{6}\,(q_1 + 1)\,(q_2 + 1)\,(q_1 + q_2 + 2)\,(2q_1 + q_2 + 3)\,,$$

given in ref. (I) with $N = 16$ we find $q_1 = 1$ and $q_2 = 1$.
The highest weight of the representation then is

$$\Lambda_1 = q_1\lambda_1 + q_2\lambda_2 = \frac{3}{2}\,a_1 + 2a_2.$$

The other weights are:

$$\Lambda_2 = \frac{3}{2} a_1 + a_2,$$

$$\Lambda_3 = \frac{1}{2} a_1 + 2a_2,$$

$$\Lambda_4 = \Lambda_5 = \frac{1}{2} a_1 + a_2 \qquad \text{(of multiplicity 2)},$$

$$\Lambda_6 = \Lambda_7 = \frac{1}{2} a_1 \qquad \text{(of multiplicity 2)},$$

$$\Lambda_8 = \frac{1}{2} a_1 - a_2,$$

$$\Lambda_9 = -\frac{1}{2} a_1 + a_2,$$

$$\Lambda_{10} = \Lambda_{11} = -\frac{1}{2} a_1, \qquad \text{(of multiplicity 2)},$$

$$\Lambda_{12} = \Lambda_{13} = -\frac{1}{2} a_1 - a_2, \qquad \text{(of multiplicity 2)},$$

$$\Lambda_{14} = -\frac{1}{2} a_1 - 2a_2,$$

$$\Lambda_{15} = -\frac{3}{2} a_1 - 2a_2,$$

$$\Lambda_{16} = -\frac{3}{2} a_1 - 2a_2.$$

The weights can be arranged in a weight diagram which we omit.

The basis elements of the Cartan subalgebra are:

$$\Gamma(\vec{h}_{a_1}) = \frac{1}{6}\,\Gamma(\vec{h}_1) = \frac{1}{6}\,\mathrm{dig}\{1,2,-1,0,0,1,1,2,-2,-1,-1,0,0,1,-2,-1\},$$

and

$$\Gamma(\vec{h}_{a_2}) = \frac{1}{12}\,\Gamma(\vec{h}_2) = \frac{1}{12}\,\mathrm{dig}\{1,-1,3,1,1,-1,-1,-3,3,1,1,-1,-1,-3,1,-1\}.$$

Constructing the differences $\Lambda_i - \Lambda_j$, $i,j = 1,2,\ldots,16$, we find that the matrix $\Gamma(\vec{e}_{a_1})$ has elements different from zero at the positions

$(p,q) = (3,1),\ (4,2),\ (5,2),\ (9,4),\ (9,5),\ (10,6),\ (10,7),\ (11,6),\ (11,7),$
$(12,8),\ (13,8),\ (15,12),\ (15,13),\ (16,14).$

The matrix $\Gamma(\vec{e}_{-a_1})$ is taken to be equal to $-\Gamma^{tr}(\vec{e}_{a_1})$. Similarly the matrix $\Gamma(\vec{e}_{a_2})$ has elements different from zero at the positions:

$(p,q) = (2,1),\ (4,3),\ (5,3),\ (6,4),\ (6,5),\ (7,4),\ (7,5),\ (8,6),\ (8,7),\ (10,9),$
$(11,9),\ (12,10),\ (12,11),\ (13,10),\ (13,11),\ (14,12),\ (14,13),\ (16,15).$

The matrix $\Gamma(\vec{e}_{-a_2})$ is taken to be equal to $-\Gamma^{tr}(\vec{e}_{a_2})$.

The matrix $\Gamma(\vec{e}_{a_1+a_2})$ is given by:

$$\Gamma(\vec{e}_{a_1+a_2}) = \frac{1}{N_{a_1,a_2}}\,[\Gamma\vec{e}_{a_1})\,,\,\Gamma(\vec{e}_{a_2})]_-,$$

The matrix $\Gamma(\vec{e}_{-a_1-a_2})$ is taken equal to $-\Gamma^{tr}(\vec{e}_{a_1+a_2})$.

The matrix $\Gamma(\vec{e}_{a_1+2a_2})$ is given by:

$$\Gamma(\vec{e}_{a_1+2a_2}) = \frac{1}{N_{a_2,a_1+a_2}}\,[\Gamma(\vec{e}_{a_2})\,,\,\Gamma(\vec{e}_{a_1+a_2})]_-\,.$$

The matrix $\Gamma(\vec{e}_{-a_1-2a_2})$ is taken equal to $-\Gamma^{tr}(\vec{e}_{a_1+2a_2})$.

Working as for the 20 - dimensional[1,2] representation we find by solving the set of simultaneous equations in the quantities e_{ij}, ϵ_{ij} that a possible solution for the ϵ_{ij} is:

$$\epsilon_{2,1} = \pm \frac{1}{2\sqrt{3}}, \quad \epsilon_{5,3} = 0, \quad \epsilon_{4,3} = \pm \frac{1}{2}, \quad \epsilon_{6,4} = 0, \quad \epsilon_{7,4} = \pm \frac{1}{\sqrt{3}},$$

$$\epsilon_{7,5} = 0, \quad \epsilon_{6,5} = \pm \frac{1}{2\sqrt{3}}, \quad \epsilon_{8,6} = 0, \quad \epsilon_{8,7} = \pm \frac{1}{2}, \quad \epsilon_{11,9} = 0,$$

$$\epsilon_{10,9} = \pm \frac{1}{2}, \quad \epsilon_{12,10} = 0, \quad \epsilon_{14,13} = \pm \frac{1}{2}, \quad \epsilon_{13,10} = \pm \frac{1}{\sqrt{3}}, \quad \epsilon_{13,11} = 0,$$

$$\epsilon_{12,11} = \pm \frac{1}{2\sqrt{3}}, \quad \epsilon_{14,12} = 0.$$

Choosing the positive sign for the ϵ_{ij} and using:

$$L_o^{16-\mathrm{dim}} = i\sqrt{3}\ \Gamma(e_{a_2}) + i\sqrt{3}\ \Gamma(e_{-a_2})$$

we find,

$$L_o^{16} = i\sqrt{3} \begin{pmatrix}
\cdot & -1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & -1/\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1/\sqrt{3} & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & -1/\sqrt{3} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/\sqrt{3} & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2\sqrt{3} & \cdot
\end{pmatrix}$$

This matrix has eigenvalues,

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{1}{2},$$

$$\lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = -\frac{1}{2},$$

$$\lambda_{13} = \lambda_{14} = \frac{3}{2}, \qquad \lambda_{15} = \lambda_{16} = -\frac{3}{2},$$

and the masses associated with the field are:

$$m_{1,2} = \frac{\chi}{\pm 3/2} = \pm \frac{2\chi}{3}, \qquad m_{3,4} = \frac{\chi}{\pm 1/2} = \pm 2\chi .$$

3. TRANSFORMATION PROPERTIES OF THE HERMITIANIZING MATRIX A

We show in this paragraph that the hermitianizing[1,2] matrix A satisfies the relation:

$$T^{+} A\, T = A, \quad (+ = \text{Hermitian conjugate}), \tag{2}$$

for every transformation of SO(4,1) which also belongs to SO(3,1) We demonstrate the validity of this relation in the case of the spin $\frac{3}{2}$ equation of Bhabha based on the 20 - dimensional representation. It is sufficient to prove it for the infinitesimal transformations. Let. $T = 1 + \in I_{ij}$ be an infinitesimal transformation which can be either a rotation on the plane (i,j) or a boost in the direction perpendicular to the plane (i,j). I_{ij} stands for the infinitesimal generators and $\in$ is the infinitesimal parameter. Expressing the generators I_{ij} by their 20 - dimensional representation matrices[1,2] $\Gamma_{20-dim}(I_{ij})$ we have:

$$T = 1 + \in \Gamma_{20-dim}(I_{ij}).$$

Substituting for T in (2) we get:

$$T^{+} A\, T = \{1 + \in \Gamma_{20-dim}(I_{ij})\}^{+} A \{1 + \in \Gamma_{20-dim}(I_{ij})\}$$

$$= \{A + \in \Gamma^{+}_{20-dim}(I_{ij})\, A\}\{1 + \in \Gamma_{20-dim}(I_{ij})\}$$

$$= A + \in \{\Gamma^{+}_{20-dim}(I_{ij})\, A + A\, \Gamma_{20-dim}(I_{ij})\} + 0\,(\in^2).$$

Thus in order that (2) holds it is necessary that :

$$\{\Gamma^{+}_{20-\dim}(I_{ij})\,A + A\,\Gamma_{20-\dim}(I_{ij})\} = 0.$$

The matrices $\Gamma_{20-\dim}(I_{ij})$ are connected to the matrices

$$\Gamma_{20-\dim}(\vec{h}_{a_1}),\ \Gamma_{20-\dim}(\vec{h}_{a_2})\ \ldots\ \Gamma_{20-\dim}(\vec{e}_{-a_1-2a_2})$$

which form the basis elements of the Lie algebra, by the relations:

(i) $\Gamma_{20-\dim}(I_{1,2}) = i\,\Gamma_{20-\dim}(\vec{h}_{a_1}) + \frac{1}{2}\,i\,\Gamma_{20-\dim}(\vec{h}_{a_2}),$

(ii) $\Gamma_{20-\dim}(I_{1,3}) = -\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{a_1}) - \frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{-a_1}) +$

$+ \frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{a_1+2a_2}) + \frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{-a_1-2a_2}),$

(iii) $\Gamma_{20-\dim}(I_{1,0}) = -\sqrt{3}\,\Gamma_{20-\dim}(\vec{e}_{a_1+a_2}) + \sqrt{3}\,\Gamma_{20-\dim}(\vec{e}_{-a_1-a_2}),$

(iv) $\Gamma_{20-\dim}(I_{2,3}) = -i\,\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{a_1}) + i\,\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{-a_1}) +$

$+ i\,\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{a_1+2a_2}) - i\,\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-\dim}(\vec{e}_{-a_1-2a_2}),$

(v) $\Gamma_{20-\dim}(I_{2,0}) = -i\sqrt{3}\,\Gamma_{20-\dim}(\vec{e}_{a_1+a_2}) + i\sqrt{3}\,\Gamma_{20-\dim}(\vec{e}_{-a_1-a_2}),$

(vi) $\Gamma_{20-\dim}(I_{3,0}) = -\sqrt{3}\,\Gamma_{20-\dim}(\vec{e}_{a_2}) + \sqrt{3}\,\Gamma_{20-\dim}(\vec{e}_{-a_2}).$

As an example, let us show how one proves the validity of $T^{+}AT = A$ when T is chosen to be:

$$T = 1 + \in \Gamma_{20-\dim}(I_{2,0}). \tag{3}$$

Using the matrix realizations of the matrices $\Gamma_{20-\dim}(\vec{e}_{a_1+a_2})$

$\Gamma_{20-dim}(e_{-a_1-a_2})^1$ we find applying (v) the matrix $\Gamma_{20-dim}(I_{2,0})$ and then by taking its complex conjugate transpose we find $\Gamma^+_{20-dim}(I_{2,0})$. Then using the hermitianizing matrix A_{20-dim} given in (1,2) we construct $\Gamma^+_{20-dim}(I_{2,0})\Lambda$ and also $\Lambda\,\Gamma_{20-dim}(I_{2,0})$. Adding these two matrices together we find that:

$$\{\Gamma^+_{20-dim}(I_{2,0})\,A + A\,\Gamma_{20-dim}(I_{2,0})\} = 0,$$

and hence $T^+ A\, T = A$ when T is chosen asin (3). Similarly we find that the same is true for all the other generators $\Gamma_{20-dim}(I_{ij})$.

4. CHARGE OF THE 16 DIMENSIONAL BHABHA FIELD

With the 16 dimensional Bhabha field one can associate the charge density $S_0 = \Psi^+ A_{16\,dim} L_0^{16-dim}\,\Psi$ where Ψ has 16 components and Ψ^+ its hermitian conjugate. The total charge is $\int S_0\, dv$. We consider that frame of reference in which $A_{16-dim}\, L_0^{16-dim}$ is diagonal, and let Λ_n, $n = 1, 2, \dots 16$ be the eigenvalues of $A_{16-dim}\, L_0^{16-dim}$ and Ψ_n the components of Ψ. Then:

$$S_0 = \Psi^+ (A_{16-dim}\, L_0^{16-dim})_{dig} \Psi = \sum_{n=1}^{16} \Lambda_n\, \Psi^*_n\, \Psi_n\,.$$

This is definite if the eigenvalues Λ_n have the same sign. To find the eigenvalues of $A_{16-dim} L_0^{16-dim}$ we use the theorem given in ref. (2) and the fact that for the 16 dimensional representation the hermitianizing matrix has the functional form:

$$A_{16-dim} = \frac{1}{3}\, L_0^{16-dim} \{4(L_0^{16-dim})^2 - 7\}$$

Thus for the eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1/2$ of L_0^{16-dim} we find:

$$\Lambda_1 = \lambda_1\, f(\lambda_1 = \frac{1}{2}) = \lambda_1 \{ \frac{1}{3}\, \lambda_1 (4\lambda_1^2 - 7)\} = -\frac{1}{2} = \Lambda_2 = \Lambda_3 = \Lambda_4 = \Lambda_5 = \Lambda_6.$$

Similarly we find for the eigenvalues

$\lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = -1/2$ of L_o^{16-dim} the eigenvalues:

$$\Lambda_7 = \lambda_7 f(\lambda_7 = -\frac{1}{2}) = -\frac{1}{2} = \Lambda_8 = \Lambda_9 = \Lambda_{10} = \Lambda_{11} = \Lambda_{12} .$$

For the eigenvalues $\lambda_{13} = \lambda_{14} = 3/2$ we find:

$$\Lambda_{13} = \lambda_{13} f(\lambda_{13} = \frac{3}{2}) = \frac{3}{2} = \Lambda_{14}$$

and for the eigenvalues $\lambda_{15} = \lambda_{16} = -3/2$ we find:

$$\Lambda_{15} = \lambda_{15} f(\lambda_{15} = -\frac{3}{2}) = \frac{3}{2} = \Lambda_{16}.$$

Hence the charge density for the 16 dimensional Bhabha field is:

$$S_0 = -\frac{1}{2} \sum_{k=1}^{12} \Psi_k^* \Psi_k + \frac{3}{2} \sum_{j=13}^{16} \Psi_j^* \Psi_j$$

which is indefinite. Thus the charge is indefinite as well.

ACKNOWLEDGEMENTS

I would like to thank Professor J.F. Cornwell of the University of St. Andrews for sugges'ing and supervising this project.

REFERENCES

1. Koutroulos, C.G., Fizika 15, 43 (1983).
2. Koutroulos, C.G., Present issue of the «Scientific Annals» of the faculty of Physics and Mathematics of the University of Thessaloniki.

ΠΕΡΙΛΗΨΗ

ΜΕΛΕΤΗ ΤΗΣ 16 - ΔΙΑΣΤΑΤΗΣ ΚΥΜΑΤΟΕΞΙΣΩΣΕΩΣ ΤΟΥ BHABHA ΚΑΙ ΤΟΥ ΦΟΡΤΙΟΥ ΤΗΣ

Ὑπὸ

Χ. Γ. ΚΟΥΤΡΟΥΛΟΥ*

Τομέας Θεωρητικῆς Φυσικῆς
Πανεπιστημίου St. Andrews
St. Andrews, Fife, SCOTLAND

Στὴν ἐργασία αὐτὴ δίδεται μία λεπτομερὴς ἔκθεση ὑπολογισμῶν ποὺ ἔχουν ἀνακοινωθεῖ προηγουμένως στὴν ἐργασία (1) καὶ ἐπιπρόσθετα ἀποδεικνύεται μὲ τὴν μέθοδον τῆς ἄλγεβρας Lie ποὺ ἔχει υἱοθετηθεῖ ἐδῶ ὅτι τὸ φορτίον τοῦ 16 - διάστατου πεδίου Bhabha εἶναι ἀπροσδιότιστον.

* Παροῦσα διεύθυνση: Σπουδαστήριο Θεωρητικῆς Φυσικῆς Πανεπιστημίου Θεσσαλονίκης.