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A STUDY OF THE 20 DIMENSIONAL BHABHA WAVE EQUATION AND ITS CHARGE

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Abstract: *A detailed exposition is given of calculations reported previously in ref. (20) and in addition it is verifield by the method adopted namely the Lie algebraic one that the charge assoccated with the 20 dimensional Bhabha field is indefinite.*

1. INTRODUCTION

The field of relativistic wave equations aims at the description of particles in terms of wave functions and equations of motion. Several wave equations have been proposed in the scientific literature, see references (1-19). Here we shall be concerned with the Bhabha [15-19] wave equation for spin $\frac{3}{2}$ particles.

Bhabha in his effort to free the higher spin theories from the presence of the subsidiary conditions proposed an equation which is similar in appearance to the Dirac wave equation and which in the absence of interactions reads

$$L_0 \frac{\partial \Psi}{\partial x_0} + L_1 \frac{\partial \Psi}{\partial x_1} + L_2 \frac{\partial \Psi}{\partial x_2} + L_3 \frac{\partial \Psi}{\partial x_3} + i x \Psi = 0 \qquad (1)$$

where L_k, $k = 0,1,2,3$ are four matrices of appropriate dimension depending on the representation according to which the wave function Ψ transforms and x a constant related to the mass of the particles.

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The Bhabha field is a multimass and multispin field. For example a field of maximum spin $\frac{3}{2}$ appears with two possible rest masses one three times the other. A field of maximum spin 2 has also two rest masses one twice the other. In a field of maximum spin s all the lower values of spin appear as well.

Bhabha studied equation (1); a) in the case when the underlying representation belongs to the group SO(4,1) and b) in the case when the underlying representation is a general representation. In the first case the matrices L_k satisfy a relation of the form:

$$[L_m, L_n]_- = I_{mn} \tag{2}$$

where I_{mn} are the infinitesimal generators of the Lorentz group. In the second case (2) does not hold. According to Bhabha there are two possible representations of the group SO(4,1) which can be used to describe a particle of spin $\frac{3}{2}$, namely the 16 and 20 dimensional representations.

In an earlier paper entitled «Lie algebras and relativistic wave equations»[20] (called in the sequel (I)) we gave a method of determining the 16 and 20 dimensional matrix representations of the group SO(4,1) in terms of which the matrices L_k of the Bhabha wave equation can be expressed. The purpose of this paper is to complete the earlier paper by providing all the missing details and also extend it by employing some of the results obtained there to attributing a charge to the spin $\frac{3}{2}$ Bhabha field. We shall concentrate here only on the 20 dimensional representation.

2. L_k AS LINEAR COMBINATIONS

The Lie algebra corresponding to SO(4,1) is the complex Lie algebra B_2[21-22]. Its generators are $\vec{h}_{a_1}$, $\vec{h}_{a_2}$, $\vec{e}_{\pm a_1}$, $\vec{e}_{\pm a_2}$, $\vec{e}_{\pm (a_1+a_2)}$, $\vec{e}_{\pm (a_1+a_2)}$.
We recall that for any Lie algebra L:

i) any element $\vec{h} \in H$ is given by

$$\vec{h} = \mu_1 \vec{h}_{a_1} + \mu_2 \vec{h}_{a_2}, \tag{3}$$

where H the Cartan subalgebra of L . $\vec{h}_{\alpha_1}$, $\vec{h}_{\alpha_2}$ the basis elements of the Cartan subalgebra and μ_1, μ_2, coefficients.

ii) $$[\vec{e}_\alpha, \vec{h}]_- = \alpha(\vec{h})\vec{e}_\alpha , \tag{4}$$

iii) $$[\vec{h}, \vec{h}']_- = 0, \ \forall \ \vec{h}, \vec{h}' \in H , \tag{5}$$

iv) $$[\vec{e}_\alpha, \vec{e}_{-\alpha}]_- = \vec{b}_\alpha, \tag{6}$$

v) $$[\vec{e}_\alpha, \vec{e}_\beta]_- = N_{\alpha,\beta}\, \vec{e}_{\alpha+\beta} , \tag{7}$$

where $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root of L. $N_{\beta,\alpha} = - N_{\alpha,\beta}$ and by convention $N_{-\alpha,-\beta} = N_{\alpha,\beta}$. Moreover if the α-string of roots containing β is:

$$\beta - r\alpha,\ \beta - (r-1)\alpha,\ \ldots \beta \ldots \beta + q\alpha \tag{8}$$

then the magnitude of $N_{\alpha,\beta}$ is given by

$$(N_{\alpha,\beta})^2 = \frac{1}{2}\, q(r+1)(\alpha,\alpha) \tag{9}$$

with the signs of $N_{\alpha,\beta}$ to some extent being arbitrary. ([,]$_-$ indicates the commutator and (,) the inner product).

Five dimensional matrix realizations of the complex Lie algebra of SO(4,1) i.e. B_2 and hence of the canonical form of B_2 are given in reference (23). These formulae for any algebra B_l are for the basis elements of the Cartan subalgebra:

$$\vec{h}_{\alpha j} = \begin{cases} -\dfrac{1}{2(2l-1)} \{\vec{e}_{j+1,j+1} - \vec{e}_{j+l+1,j+l+1} - \vec{e}_{j+2,j+2} + \vec{e}_{j+l+2,j+l+2}\}, j=1,2,\ldots(l-1), \\ -\dfrac{1}{2(2l-1)} \{\vec{e}_{l+1,l+1} - \vec{e}_{2l+1,2l+1}\} ,\ i = l , \end{cases} \tag{10}$$

and for the other elements corresponding to the simple roots:

$$\vec{e}_{\alpha j} = \begin{cases} [2(2l-1)]^{-1/2} \{\vec{e}_{j+1,j+2} - \vec{e}_{j+l+2,j+l+1}\} ,\ i = 1,2,\ldots(l-1), \\ [2(2l-1)]^{-1/2} \{\vec{e}_{1,2l+1} - \vec{e}_{l+1,1}\} ,\quad j = l, \end{cases} \tag{11}$$

$$e_{-a_j} = \begin{cases} -[2(2l-1)]^{-1/2}\{\vec{e}_{j+2,j+1} - \vec{e}_{j+l+1,j+l+2}\}, & i = 1,2,\ldots(l-1), \\ -[2(2l-1)]^{-1/2}\{\vec{e}_{2l+1,1} - \vec{e}_{1,l+1}\}, & j = l. \end{cases} \quad (12)$$

In the above $\vec{e}_{m,n}$ are square matrices of appropriate dimension in which the (m,n) element is unit and all the other elements are zero. The inner product of a simple root a_j by itself, for the algebra B_l is given by the formula:

$$(a_j,a_j) = \begin{cases} \dfrac{1}{2l-1}, & j = 1,2,\ldots(l-1) \\ \dfrac{1}{2(2l-1)}, & j = l. \end{cases} \quad (13)$$

In the case of the algebra B_2 (i.e. $l = 2$) for $j = 1$, $l = 2$ we find formula (9) and for $j = 2$, $l = 2$ we find formula (10) of ref. (I). Likewise for $j = 1$, $l = 2$ we find formulae (11) and (12) of ref. (I) and for $j = 2$, $l = 2$ we find formulae (13) and (14) of the same reference. For the inner products we find for $j = 1$, $l = 2$, $(a_1,a_1) = \frac{1}{3}$ and for $j = 2$, $l = 2 (a_2, a_2) = \frac{1}{6}$.

Here we would like to point out that some authors define as basis elements of the Cartan subalgebra instead of $\vec{h}_{a_j}$ the elements $\vec{h}_j$ related to $\vec{h}_{a_j}$ by $\vec{h}_{a_j} = \frac{(a_j,a_j)}{2}\vec{h}_j$. Thus in the case of B_2, $\vec{h}_{a_1} = \frac{1}{6}\vec{h}_1$ and $\vec{h}_{a_2} = \frac{1}{12}\vec{h}_2$. For $\vec{e}_{a_j}$ they use $\vec{e}_j$ related as follows $\vec{e}_{a_j} \equiv \vec{e}_j$ and for $\vec{e}_{-a_j}$ they use $\vec{f}_j$ connected by $\vec{f}_j = 2\vec{e}_{-a_j}/(a_j,a_j)$. In the following we shall distinguish the matrices associated with the elements

$\vec{h}_{a_1}$, $\vec{h}_{a_2}$, $\vec{e}_{\pm a_1}$, $\vec{e}_{\pm a_2}$ ... by writing $\Gamma(\vec{h}_{a_1})$, $\Gamma(\vec{h}_{a_2})$, $\Gamma(\vec{e}_{\pm a_1})$, $\Gamma(\vec{e}_{\pm a_2})$....

To determine the basis elements $\vec{e}_{(a_1+a_2)}$, $\vec{e}_{-(a_1+a_2)}$, $\vec{e}_{(a_1+2a_2)}$, $\vec{e}_{-(a_1+2a_2)}$ of B_2 in the five dimensional representation we make use of the following formulae:

$$\vec{e}_{(a_1+a_2)} = \frac{1}{N_{a_1,\,a_2}}[\vec{e}_{a_1}, \vec{e}_{a_2}]_- \;,\; \vec{e}_{-(a_1+a_2)} = \frac{1}{N_{-a_1-a_2}}[\vec{e}_{-a_1}, \vec{e}_{-a_2}]_-$$

$$\vec{e}_{(a_1+2a_2)} = \frac{1}{N_{a_2,a_1+a_2}}[\vec{e}_{a_2}, \vec{e}_{a_1+a_2}] \;,\; \vec{e}_{-(a_1+2a_2)} = \frac{1}{N_{-a_2,\,-a_1-a_2}}[\vec{e}_{-a_2}, \vec{e}_{-a_1-a_2}]_-$$

or in matrix notation:

$$\Gamma(\vec{e}_{a_1+a_2}) = \frac{1}{N_{a_1,a_2}}[\Gamma(\vec{e}_{a_1}),\Gamma(\vec{e}_{a_2})], \Gamma(e_{-a_1-a_2}) = \frac{1}{N_{-a_1-a_2}}[\Gamma(\vec{e}_{-a_1}),\Gamma(\vec{e}_{-a_2})]_-,$$

$$\Gamma(\vec{e}_{a_1+2a_2}) = \frac{1}{N_{a_2,a_1+a_2}}[\Gamma(\vec{e}_{a_2}), \Gamma(\vec{e}_{a_1+a_2})]_- ,$$

$$\Gamma(\vec{e}_{-a_1-2a_2}) = \frac{1}{N_{-a_2,-a_1-a_2}}[\Gamma(\vec{e}_{-a_2}),\Gamma(\vec{e}_{-a_1-a_2})]_-.$$

Thus using $\Gamma(\vec{e}_{a_1})$, $\Gamma(\vec{a}_2)$, $\Gamma(\vec{e}_{-a_1})$, $\Gamma(\vec{e}_{-a_2})$ and $N_{a_1,a_2} = N_{-a_1,-a_2} = \pm\frac{1}{\sqrt{6}}$ (to be determined later) we find choosing the positive sign the formulae (15) and (16) of ref. (I). Similarly using $\Gamma(\vec{e}_{a_2})$, $\Gamma(\vec{e}_{a_1+a_2})$, $\Gamma(\vec{e}_{-a_1})$, $\Gamma(\vec{e}_{-a_1-a_2})$ and $N_{a_2,a_1+a_2} = N_{-a_2,-a_1-a_2} = \pm\frac{1}{\sqrt{6}}$ we find choosing the positive sign the formulae (17) and (18) of the same reference.

To determine N_{a_1,a_2} we use the formula:

$$(N_{a_1,a_2})^2 = \frac{1}{2}\,q(r+1)\,(a_1,a_1)$$

where q, r are determined from the a_1 - series of roots containing a_2 which is $\{a_2, a_2 + 1a_1\}$ and hence $r = 0$, $q = 1$. Using $(a_1,a_1) = 1/3$ we find $N_{a1,a2} = \pm 1\sqrt{6}$. Similarly to determine N_{a_2,a_1+a_2}, we use the formula $(N_{a_2,a_1+a_2})^2 = (1/2)q(r+1)(a_2,a_2)$ where q, r are now determined from the a_2 series of roots containing $a_1 + a_2$ which is $\{(a_1 + a_2) - a_2, (a_1 + a_2), (a_1 + a_2) + a_2\}$ and hence $r = 1$, $q = 1$. Using $(a_2,a_2) = 1/6$ we find $N_{a_1,a_2+a_1} = \pm 1\sqrt{6}$.

Bhabha in defining the five dimensional realizations of his matrices L_k, $k = 0,1,2,3$ extended the group SO(3,1) to the group SO(4,1) by identifying:

$$L_0 = I_{0,4}\,,\ L_1 = I_{1,4}\,,\ L_2 = I_{2,4}\,,\ L_3 = I_{3,4}$$

where $I_{0,4}$, $I_{1,4}$, $I_{2,4}$, $I_{3,4}$ are generators of the five dimensional Lorentz group. To derive the generators I_{ij} we consider a rotation $R_{ij}(\varphi)$ through an angle φ in the (i,j) plane and identify the generator I_{ij} with the derivative of $R_{ij}(\varphi)$ with respect to φ at $\varphi = 0$ i.e.:

$$I_{ij} = \frac{d}{d\varphi} R_{ij}(\varphi)\Big|_{\varphi = 0}$$

There is a similarity transformation S which maps the canonical form of B_l to SO $(2l + 1 - 2r, 2r)$, $r = 0,1,\ldots l$ and is defined in the following theorem[23].

Theorem: Let b be an element of the matrix realization of the canonical form L. Then the similarity transformation to the SO$(2l + 1 - 2r, 2r)$ Lie algebra (for $r = 0, 1, \ldots, l$) is given by $a = SbS^{-1}$ where $S = \sqrt{g}\, T$ where T is given by:

$$T_{jk} = \begin{cases} 1 \,, & j = 2k - 2,\ k = 2,\ldots, l + 1 \\ & \text{and } j = 2k - 2l - 2,\ k = l + 2,\ldots, 2l + 1 \\ i \,, & j = 2k - 3,\ k = 2,\ldots, l + 1 \\ -i \,, & j = 2k - 2l - 3,\ k = l + 2,\ldots, 2l + 1 \\ \sqrt{2} \,, & j = 2l + 1,\ k = 1 \\ 0 \,, & \text{all other } j, k \end{cases}$$

provided that the diagonal elements of g of $a^{tr}g + ga = 0$, (tr = transpose) are arranged so that $g_{2j,2j} = g_{2j-1,2j-1}$ for $j = 1,2,\ldots l$ and $g_{2j+1,2j+1} = g_{2j-1,2j-1}\exp\{a_j(\vec{h})\}$ for $j = 1,2,\ldots,l$ where $\exp\{a_j(\vec{h})\}$ are given by:

$$\exp\{a_j(\vec{h})\} = \begin{cases} 1 \,, & j = 1,2,\ldots l - 1,\ j \neq l - r \\ -1 \,, & j = l - r,\ l \end{cases}$$

where $a_1, a_2, \ldots, a_l$ are the simple roots of B_l.
It will be noted that the only dependence of S on the elements of g lies in the factor $\sqrt{g}$ which is defined to he the diagonal matrix such that:

$$(\sqrt{g})_{jj} = \begin{cases} 1 & \text{if } g_{jj} = 1, \\ i & \text{if } g_{jj} = -1. \end{cases}$$

Setting $l = 2$, $r = 2$ in the above theorem we have a mapping of the canonical form of the Lie algebra B_2 onto SO(4,1). In this case:

$$T = \begin{pmatrix} 0 & i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & i & 0 & -i \\ 0 & 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$g = \mathrm{dig}(1,1,1,1,-1)$ and $\sqrt{g} = \mathrm{dig}(1,1,1,1,i)$. Hence we find for the similarity transformation S which maps the canonical form of B_2 to the Lie algebra SO(4,1) the matrix (20) of ref. (I). The inverse of S is:

$$S^{-1} = \frac{\mathrm{AdjS}}{\mathrm{detS}} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/2 & 0 & 0 & 0 \\ 0 & 0 & -i/2 & 1/2 & 0 \\ i/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & i/2 & 1/2 & 0 \end{pmatrix}$$

Using the similarity transformation S and constructing $S^{-1}L_kS$, $k = 0,1,2,3$ we find the matrices (21) given in ref. (I). The matrices $S^{-1}L_kS$ form a linear combination of the basis elements of the Lie algebra B_2, as indicated by formula (22) of ref. (I). As an example we find the coefficients of the linear combination giving $S^{-1}L_0S$. We have:

$$S^{-1}L_0S = \begin{bmatrix} 0 & , \theta/\sqrt{6} & , \zeta/\sqrt{6} & , \eta/\sqrt{6} & , \varepsilon/\sqrt{6} \\ -\eta/\sqrt{6} & , -\alpha/\sqrt{6} & , \gamma/\sqrt{6} & , 0 & , \varkappa/\sqrt{6} \\ -\varepsilon/\sqrt{6} & , -\delta/\sqrt{6} & , \frac{\alpha}{6} - \frac{\beta}{6} & , -\varkappa/\sqrt{6} & , 0 \\ -\theta/\sqrt{6} & , 0 & , \lambda/\sqrt{6} & , \alpha/6 & , \delta/\sqrt{6} \\ -\zeta/\sqrt{6} & , -\lambda/\sqrt{6} & , 0 & , -\gamma/\sqrt{6} & , -\frac{\alpha}{6} + \frac{\beta}{6} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \\ -i/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i/\sqrt{2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equating corresponding elements we find $\zeta = i\sqrt{3}$ and $\varepsilon = i\sqrt{3}$ with all the other coefficients zero. Thus we find for the linear combination giving $S^{-1}L_0S$ the formula (23) of ref. (I). Similarly working we find for the other three linear combinations the formulae (24), (25) and (26), of the same ref.

The 16 and 20 dimensional realizations of the matrices L_k are give by the same linear combinations of the basis elements of B_2 as for the 5 dimensional representation except that $\Gamma_{5\text{-dim}}(\vec{h}_{a_1}), \ldots \Gamma_{5\text{-dim}}(\vec{e}_{-a_1-2a_2})$ have to be replaced by the 16 or 20 dimensional matrices $\Gamma_{20\text{-dim}}(\vec{h}_{a_1}), \ldots,$ $\Gamma_{20\text{-dim}}(\vec{e}_{-a_1-2a_2})$, as is indicated by formulae (27), (28), (29) and (30) of the ref. (I). Hence for the determination of the matrices L_k in the 20 dimensional representation the basis elements $\Gamma_{20\text{-dim}}(\vec{h}_{a_1})$, $\Gamma_{20\text{-dim}}$ $(\vec{h}_{a_2}) \ldots$ $\Gamma_{20\text{-dim}}$ $(\vec{e}_{-a_1-2a_2})$ of B_2 are necessary. These we give in tbe following paragraph.

3. 20 - DIMENSIONAL BASIS OF B_2 ; THE MATRICES $L_k^{20\text{-dim}}$; THE HERMITIANIZING MATRIX; THE EIGENVALGES OF L_0

We give now here the 20 dimensional basis elements of B_2. The fundamental weights of B_2 are given by formula (31) of ref. ( ) and the highest weight is given by formula (32) while the dimension of the representation is given by formula (33) of ref. (I). For the 20 dimensional representation the highest weight is given by formula (34). The other weights of the representation are found as follows We start with the highest weight $\Lambda = (3/2)a_1 + 3a_2$ and evaluate the ratios:

$$\frac{2(\Lambda,a_1)}{(a_1, a_1)} \quad , \quad \frac{2(\Lambda,a_2)}{(a_2, a_2)} \quad .$$

Evaluation of the ratio $2(\Lambda,a_1) / (a_1,a_1)$.

We have:

$$\frac{2(\Lambda,a_1)}{(a_1,a_1)} = \frac{2(3/2a_1 + 3a_2,a_1)}{(a_1, a_1)} = \frac{2((3/2a_1,a_1) + (3a_2,a_1))}{(a_1, a_1)}$$

$$= \frac{2(3/2(a_1, a_1) + 3(a_2, a_1))}{(a_1, a_1)} \quad .$$

For the evaluation of the ratio the values of (a_1,a_1) and (a_2,a_1) are necessary. (a_1,a_1) is 1/3. To find (a_2,a_1) we make use of the Cartan matrix B_2 defined by:

$$A_{jk} = \frac{2(a_j, a_k)}{(a_j,a_j)'} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} .$$

Setting $A_{21} = 2(a_2,a_1) / (a_2,a_2) = -2$ we find using $(a_2,a_2) = 1/6$ that $(a_2,a_1) = -1/6$. Thus $2(\Lambda,a_1) / (a_1,a_1) = 0$ and the a_1 series of weights containing Λ is just $\Lambda = 3/2a_1 + 3a_2$.

Evaluation of the ratio $2(\Lambda,a_2) / (a_2,a_2)$

We have:

$$\frac{2(\Lambda\ a_2)}{(a_2,a_2)} = \frac{2(3/2a_1 + 3a_2,a_2)}{(a_2,a_2)} = \frac{2(3/2(a_1,a_2) + 3(a_2,a_2))}{(a_2,a_2)} \; .$$

Using (a_2,a_2) and $(a_1,a_2) = -1/6$ We find $2(\Lambda,a_2) / (a_2,a_2) = 3 \equiv j - k$. Where j is the lower integer and k the upper integer in the a_2 - series of weights containing Λ i.e.:

$$\{\Lambda - ja_2, \Lambda - (j-1)a_2, \ldots \Lambda + (k-1)a_2, \Lambda + ka_2\}$$

The equation $j - k = 3$ accepts the solution $k = 0$, $j = 3$ and hence the a_2 - series of weights containing Λ is:

$$\{\Lambda - 3a_2, \Lambda - 2a_2, \Lambda - a_2, \Lambda\}$$

or

$$\{3/2a_1 + 3a_2, 3/2a_1 + 2a_2, 3/2a_1 + a_2, 3/2a_1\}.$$

Choosing any weight of the above set for example $3/2a_1$ and proceeding in the same way as for the highest weight we find some new weights of the representation. Continuing in the same way with every new weight we find finally the following sixteen weights:

$\Lambda \equiv M_1 = 3/2a_1 + 3a_2$, $M_2 = 3/2a_1 + 2a_2$, $M_3 = 3/2a_1 + a_2$, $M_4 = 3/2a_1$,

$M_5 = 1/2a_1 + 2a_2$, $M_6 = 1/2a_1 + a_2$, $M_7 = 1/2a_1$, $M_8 = 1/2a_1 - a_2$,

$M_9 = - 1/2a_1 + a_2$, $M_{10} = - 1/2a_1$, $M_{11} = - 1/2a_1 - a_2$, $M_{12} = - 3/2a_1$,

$M_{13} = - 1/2a_1 - 2a_2$, $M_{14} = - 3/2a_1 - a_2$, $M_{15} = - 3/2a_1 - 2a_2$,

$M_{16} = - 3/2a_1 - 3a_2$.

Because the number of the weights found is less than the degree of the representation some weights are multiple. In general the method of calculating the multiplicity n_M of a weight M is given by Freudentals[22] recursion formula:

$$\{(\Lambda + \delta, \Lambda + \delta) - (M + \delta, M + \delta)\}n_M = 2 \sum_{k=1,a>0}^{\infty} \sum n_{M+ka}(M + ka, a)$$

where $\delta = 1/2 \sum_{a>0} a$, and the sum is taken over all the positive roots.

In applying Freudentals formula we have to bear in mind that the multiplicity of the highest weight is always one i.e. $n_\Lambda = 1$.

In the case of B_2 the positive roots are a_1, a_2, $(a_1 + a_2)$, $(a_1 + 2a_2)$ and δ is:

$$\delta = \frac{1}{2}\{a_1 + a_2 + (a_1 + a_2) + (a_1 + 2a_2)\} = \frac{3}{2} a_1 + 2a_2$$

As an example let us find the multiplicity of the weight $M_2 = 3/2a_1 + 2a_2$. Substituting in Freudental's formula $\Lambda = 3/2a_1 + 3a_2$, $M = M_2 = 3/2a_1 + 2a_2$, $n_M = n_{M_2}$, $M + ka = M_2 + ka$, $\delta = 3/2a_1 + 2a_2$ the left hand side gives:

$$\{(\Lambda + \delta, \Lambda + \delta) - (M_2 + \delta, M_2 + \delta)\}n_{M_2} =$$

$$\{3(a_1,a_2) + 3(a_2,a_1) + 9(a_2,a_2)\}n_{M_2} = \frac{1}{2} n_{M_2} .$$

The right hand side using $n_\Lambda = 1$ gives:

$$2 \sum_{a>0} \sum_{k=1}^{1} n_{M_2+ka}(M_2 + ka,a) = 2 \sum_{a>0} n_{M_2+a_2}(M_2 + a_2,a_2) =$$

$$= 2n_\Lambda(\frac{3}{2} a_1 + 3a_2,a_2) = 2(\frac{3}{2} (a_1,a_2) + 3(a_2,a_2)) = \frac{1}{2} .$$

Equating the left and right hand sides we find $n_{M_2} = 1$. Similarly we find for the other multiplicities:

$n_\Lambda = 1$, $n_{M_3} = 1$, $n_{M_4} = 1$, $n_{M_5} = 1$, $n_{M_6} = 2$, $n_{M_7} = 2$, $n_{M_8} = 1$, $n_{M_9} = 1$,

$n_{M_{10}} = 2$, $n_{M_{11}} = 2$, $n_{M_{12}} = 1$, $n_{M_{13}} = 1$, $n_{M_{14}} = 1$, $n_{M_{15}} = 1$, $n_{M_{16}} = 1$.

Taking into consideration the multiplicities of the weights and renaming them as Λ_j, $j = 1,2,\ldots 20$ we find the twenty weights given by formulae (35) of ref. (I). These weights can be arranged in a weight diagram which we omit. Using the test of reflections we can check that the above weights are all the weights of the representation. We use as lines of reflection the lines perpendicular to the roots a_1, a_2, $a_1 + a_2$, $a_1 + 2a_2$. The test of reflections states that for every non-zero root a relative to the space of roots a linear transformation S_a in the linear space of weights is defined such that:

$$S_a(\Lambda_i) = \Lambda_i - \frac{2(\Lambda_i,a)}{(a,a)} a ,$$

for any weight Λ_i. As an example let us find the reflection of the highest weight $\Lambda = 3/2a_1 + 3a_2$ through the line perpendicular to the root $a = a_1 + 2a_2$. We have:

$$S_a(\Lambda)=(\frac{3}{2}a_1+3a_2)-\frac{2(3/2a_1+3a_2,a_1+2a_2)}{(a_1+2a_2,a_1+2a_2)}(a_1+2a_2)=-\frac{3}{2}a_1-3a_2=\Lambda_{20}.$$

Basis of the Cartan subalgebra:

We find now the matrices $\Gamma(\vec{h}_{a_1})$, $\Gamma(\vec{h}_{a_2})$ forming the basis of the Cartan subalgebra. In doing so we find first the matrices $\Gamma(\vec{h}_1)$, $\Gamma(\vec{h}_2)$.

We require: a) $\Gamma(\vec{h}_j)_{pp} = \Lambda_p(\vec{h}_j)$, $j = 1,2$, $p = 1,2,\ldots 20$ where Λ_p is the p-th weight of the representation and b) $a_k(\vec{h}_j) = A_{jk}$ where A_{jk} are the elements of the Cartan matrix B_2.

i) $\Gamma(\vec{h}_{a_1})$: The elements of $\Gamma(h_1)$ are calculated as follows:

$$\Gamma(\vec{h}_1)_{1,1}=\Lambda_1(\vec{h}_1)=\frac{3}{2}a_1(\vec{h}_1)+3a_2(\vec{h}_1)=\frac{3}{2}A_{1,1}+3A_{1,2}=\frac{3}{2}(2)+3(-1)=0.$$

Similarly working we find the remaining diagonal matrix elements. The off diagonal elements are zero. Thus

$$\Gamma(\vec{h}_1) = \mathrm{dig}\{0,1,2,3,-1,0,0,1,1,2,-2,-1,-1,0,0,-3,1,-2,-1,0\}$$

which is related to $\Gamma(h_{a_1})$ by the formula $\Gamma(h_{a_1}) = 1/6\Gamma(\vec{h}_1)$, (i.e. we find (37) of ref. (I)).

ii) $\Gamma(\vec{h}_{a_2})$: The elements of $\Gamma(\vec{h}_2)$ are calculated as follows:

$$\Gamma(\vec{h}_2)_{1,1}=\Lambda_1(\vec{h}_2)=\frac{3}{2}a_1(\vec{h}_2)+3a_2(\vec{h}_2)=\frac{3}{2}A_{2,1}+3A_{2,2}=\frac{3}{2}(-2)+3(2)=3.$$

Similarly working we find the remaining diagonal elements. The off diagonal elements are zero. Thus:

$$\Gamma(\vec{h}_2) = \mathrm{dig}\{3,1,-1,-3,3,1,1,-1,-1,-3,3,1,1,-1,-1,3,-3,1,-1,-3\},$$

and $\Gamma(\vec{h}_{a_2}) = \frac{1}{12}\Gamma(\vec{h}_2)$ (i.e. (38) of ref. (I)).

We remark that $\Gamma(\vec{h}_{a_1})$ and $\Gamma(\vec{h}_{a_2})$ are traceless.

The other basis elements of B_2:

By definition $[\Gamma(\vec{h}), \Gamma(\vec{e}_a)] = -a(\vec{h})\Gamma(\vec{e}_a)$. From this we have $\{\Gamma(\vec{h})_{pp} - \Gamma(\vec{h})_{qq} + a(\vec{h})\}\Gamma(\vec{e}_a)_{pq} = 0$. The matrix element $\Gamma(\vec{e}_a)_{pq} \neq 0$ only if $\Gamma(\vec{h})_{pp} - \Gamma(\vec{h})_{qq} = -a(\vec{h})$, or if the difference between the p-th weight and the q-th weight is equal to $-a(\vec{h})$ i.e. $\Lambda_p(\vec{h}) - \Lambda_q(\vec{h}) = -a(\vec{h})$. These differences give the positions (p,q) at which the matrix $\Gamma(\vec{e}_a)$ has elements different from zero.

i) $\Gamma(\vec{e}_{a_1})$: Using the weights $\Lambda_1, \Lambda_2, \ldots \Lambda_{20}$ found earlier and constructing the differences $\Lambda_p(\vec{h}) - \Lambda_q(\vec{h})$ and selecting out of them those for which the relation $\Lambda_p(\vec{h}) - \Lambda_q(\vec{h}) = -a_1(\vec{h})$ is satisfied we find that the matrix $\Gamma(\vec{e}_{a_2})$ has non-zero elements at the positions (p,q) given by (39) of ref. (I). We call these elements $e_{5,2}$, $e_{6,3}$, $e_{7,3}$, $e_{8,4}$, $e_{9,4}$, $e_{11,6}$, $e_{11,7}$, $e_{12,8}$, $e_{12,9}$, $e_{13,8}$, $e_{13,9}$, $e_{14,10}$, $e_{15,10}$, $e_{16,12}$, $e_{16,13}$, $e_{18,14}$, $e_{18,15}$, $e_{19,17}$.

ii) $\Gamma(\vec{e}_{-a_1})$: We choose $\Gamma(\vec{e}_{-a_1}) = -\Gamma^{tr}(\vec{e}_{a_1})$ Thus the non-zero elements of $\Gamma(\vec{e}_{-a_1})$ appear at the transposed positions of those of $\Gamma(\vec{e}_{a_1})$

iii) $\Gamma(\vec{e}_{a_2})$: Constructing again the differences $\Lambda_p(\vec{h}) - \Lambda_q(\vec{h})$ and selecting those for which $\Lambda_p(\vec{h}) - \Lambda_q(\vec{h}) = -a_2(\vec{h})$ is satisfied we find that the matrix $\Gamma(\vec{e}_{a_2})$ has non-zero elements at the positions (p,q) given by (40) of ref. (I). We call these elements $\varepsilon_{2,1}$, $\varepsilon_{3,2}$, $\varepsilon_{4,3}$, $\varepsilon_{6,5}$, $\varepsilon_{7,5}$, $\varepsilon_{8,6}$, $\varepsilon_{8,7}$, $\varepsilon_{9,6}$, $\varepsilon_{9,7}$, $\varepsilon_{10,8}$, $\varepsilon_{10,9}$, $\varepsilon_{12,11}$, $\varepsilon_{13,11}$, $\varepsilon_{14,12}$, $\varepsilon_{14,13}$, $\varepsilon_{15,12}$, $\varepsilon_{15,13}$, $\varepsilon_{17,14}$, $\varepsilon_{17,15}$, $\varepsilon_{18,16}$, $\varepsilon_{19,18}$, $\varepsilon_{20,19}$.

iv) $\Gamma(\vec{e}_{-a_2})$: We choose $\Gamma(\vec{e}_{-a_2}) = -\Gamma^{tr}(\vec{e}_{a_2})$. Thus the non-zero elements of $\Gamma(\vec{e}_{-a_2})$ are at the transposed positions of those of $\Gamma(\vec{\epsilon}_{a_2})$.

v) $\Gamma(\vec{e}_{a_1+a_2})$: This matrix is given by formula (41) of ref. (I).

vi) $\Gamma(\vec{e}_{-a_1-a_2})$: This matrix is given by $\Gamma(\vec{e}_{-a_1-a_2}) = -\Gamma^{tr}(\vec{e}_{a_1+a_2})$.

vii) $\Gamma(\vec{e}_{a_1+2a_2})$: This matrix is given by (42) of ref. (I).

viii) $\Gamma(\vec{e}_{-a_1-2a_2})$: This matrix is given by $\Gamma(\vec{e}_{-a_1-2a_2}) = -\Gamma^{tr}(\vec{e}_{a_1+2a_2})$.

All the basis elements $\Gamma(\vec{e}_{+a_1})$, $\Gamma(\vec{e}_{\pm a_2})$, $\Gamma(\vec{e}_{\pm(a_1+a_2)})$ and $\Gamma(\vec{e}_{\pm(a_1+2a_2)})$ are functions of the quantities e_{ij}, ε_{ij}. To determine these quantities we make use of the commutation relations of the Lie algebra, namely: (43), (44), (45), (46) and (47) of ref. (I). From each one of these relations we derive a set of equations satisfied by the quantities e_{ij}, ε_{ij}. Solving these equations simultaneously we find the solution given by (48) and (49) of ref. (I). Choosing for the ε_{ij} the positive sign this fixes the sign of e_{ij} to positive. Hence we have for the basis matrices of B_2 the matrices given by (50), (51), (52), (53), (54), (55), (56) and (57). By means of these matrices the matrices L_k^{20-dim}, $(k = 0,1,2,3)$ of the Bhabha wave equation can be constructed as is indicated and given explicitly by (58), (59), (60) and (61) of ref. (I). The hermitianizing matrix A_{20-dim} is given by (71) of ref. (I). The eigenvalues of L_0^{20-dim} are given by (74) and the masses associated with the field are given by (75) of ref. (I).

We shall use in the following paragraph the results referred to here to study the charge of the 20 dimensional Bhabha wave equation.

3. CHARGE OF THE 20 DIMENSIONAL BHABHA WAVE EQUATION

With every Bhabha wave equation one can associate the quantity S_0 known as the charge density and given by the formula $S_0 = \Psi^+ A L_0 \Psi$ where Ψ is a vector with the same number of components as the dimension of the representation, Ψ^+ its hermitian conjugate. The total charge is given by $\int S_0 dv$ where dv the volume element.

Since the charge is independent of the frame of reference it is better to consider that frame in which the matrix AL_0 is diagonal. If Λ_n, $n = 1,2,\ldots 20$ are the eigenvalues of AL_0 and Ψ_n the components of Ψ then:

$$S_0 = \Psi^+ (AL_0)_{diag} \, \Psi = \sum_{n=1}^{20} \Lambda_n \Psi_n^* \Psi_n \, .$$

This is definite if the eigenvalues Λ_n have the same sign. To find the eigenvalues of AL_0 we use the following theorem. If $A = f(L_0)$ and λ is an eigenvalue of L_0 then $f(\lambda)$ is an eigenvalue of A and $\lambda f(\lambda) = \Lambda$ is an eigenvalue of AL_0. In the case of the spin 3/2 field associated with the 20 dimensional representation A_{20-dim} has the functional form:

$$A_{20-dim} = f(L_o^{20-dim}) = \frac{1}{3} L_o^{20-dim}\{4(L_o^{20-dim})^2 - 7\}$$

and for the eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 3/2$ of L_o^{20-dim} we find:

$$f(\lambda_1 = 3/2) = \frac{1}{3} \lambda_1(4\lambda_1^2 - 7) = 1 = f(\lambda_2) = f(\lambda_3) = f(\lambda_4)$$

Thus the corresponding eigenvalues of $A_{20-dim}\, L_o^{20-dim}$ are:

$$\Lambda_1 = \lambda_1 f(\lambda_1) = \frac{3}{2} = \Lambda_2 = \Lambda_3 = \Lambda_4.$$

Similarly for the other eigenvalues of L_o^{20-dim} we find that the corresponding eigenvalues of $A_{20-dim}\, L^{20-dim}$ are:

$$\Lambda_5 = \lambda_5 f(\lambda_5 = -3/2) = -\frac{3}{2}(-1) = \frac{3}{2} = \Lambda_6 = \Lambda_7 = \Lambda_8$$

$$\Lambda_9 = \lambda_9 f(\lambda_9 = 1/2) = \frac{1}{2}(-1) = -\frac{1}{2} = \Lambda_{10} = \Lambda_{11} = \Lambda_{12} = \Lambda_{13} = \Lambda_{14}$$

$$\Lambda_{15} = \lambda_{15} f(\lambda_{15} = -1/2) = -\frac{1}{2}(1) = -\frac{1}{2} = \Lambda_{16} = \Lambda_{17} = \Lambda_{18} = \Lambda_{19} = \Lambda_{20}.$$

Substituting in $S_0 = \Psi^+ \ (A_{20-dim}\, L_o^{20-dim})_{dig}\, \Psi$ we find for the charge density:

$$S_0 = \frac{3}{2} \sum_{j=1}^{8} \Psi_j^* \Psi_j - \frac{1}{2} \sum_{k=9}^{20} \Psi_k^* \Psi_k .$$

This is not definite and hence the charge is not definite.

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REFERENCES

1. Schrodinger, E., Ann. der Phys. 79, 361 (1926).
2. Schrodinger, E., Ann. der Phys. 79, 489 (1926).
3. Schrodinger, E., Ann. der Phys. 80, 437 (1926).
4. Schrodinger, E., Ann. der Phys. 81, 109 (1926).
5. Gordon, N., Z. fur Physik, 40, 117 (1926).
6. Klein, O., Z. fur Physik, 41, 407 (1927).
7. Dirac, P.A.M., Proc. Roy. Soc. 117, 610 (1928).
8. Dirac, P.A.M., Proc. Roy. Soc. 118, 351 (1928).
9. Dirac, P.A.M., Proc. Roy. Soc. A 155, 447 (1936).
10. Fierz M., Helv. Phys. Acta, 12, 3 (1939).
11. Fierz, M. and Pauli, W., Proc. Roy. Soc. A 173, 211 (1939).
12. Kemmer, N., Proc. Roy. Soc. A 173, 91 (1939).
13. Duffin, R.J., Phys. Rev. 54, 1114 (1939).
14. Rarita, W. and Schwinger, J., Phys. Rev. 60, 61 (1941).
15. Bhabha, H.J., Proc. Indian Acad. Sci. A 21, 241 (1945).
16. Bhabha, H.J., Revs. Modern Phys. 17, 200 (1945).
17. Bhabha, H.J., Revs. Modern Phys. 21, 451 (1945).
18. Bhabha, H.J., Proc. Indian Acad. Sci. A 34, 335 (1951).
19. Bhabha, H.J., Phil. Mag. 43, 33 (1952).
20. Koutroulos, C.G., Fizika 15, 43 (1983).
21. Konuma, M., Shima, K. and Wada, W., Progr. Theor. Phys. Suppl. 28, 1 (1963).
22. Jackobson, N., «Lie Algebras» Interscience publishers (John Wiley and sons) New York (1966).
23. Cornwell, J.F., Inter. J. Theor. Phys. 12, 333 (1975).

ΠΕΡΙΛΗΨΗ

ΜΕΛΕΤΗ ΤΗΣ 20 - ΔΙΑΣΤΑΤΗΣ ΚΥΜΑΤΟΕΞΙΣΩΣΕΩΣ BHABHA ΚΑΙ ΤΟΥ ΦΟΡΤΙΟΥ ΤΗΣ

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Στὴν ἐργασία αὐτὴ δίδεται μία λεπτομερὴς ἔκθεση ὑπολογισμῶν πoὺ ἔχουν ἀνακοινωθεῖ προηγουμένως στὴν ἐργασία (22) καὶ ἐπιπρόσθετα ἀποδεικνύεται μὲ τὴν μέθοδον τῆς ἄλγεβρας Lie πoὺ ἔχει υἱοθετηθεῖ ἐδῶ ὅτι τὸ φορτίον τοῦ 20 - διάστατου πεδίου Bhabha εἶναι ἀπροσδιόριστον.

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