Sci. Annals., Fac. Phys. & Mathem., Univ. Thessaloniki, **19**, 175, (1979)

AN EXTENSION OF WASSCHER'S METHOD FOR GALVANOMAGNETIC MEASUREMENTS ON ANISOTROPIC MATERIALS

by
D. S. KYRIAKOS and N. A. ECONOMOU
(Department of Physics, University of Thessaloniki)
(*Received* 21.11.79)

Abstract: *An appropriate computer programme is used to simplify the calculations in the case of Wasscher's method for galvanomagnetic measurements in anisotropic materials. For flat circular samples the main anisotropy ratio of the in plane resistivity is determined and suitable formulas for the principal resistivities are derived.*

1. INTRODUCTION

In 1958 Van der Pauw introduced a new method for measuring galvanomagnetic coefficients [1]. This method is based upon conformal mapping and it can be applied to isotropic materials and flat samples of arbitrary shape. Ten years later the method was extended for anisotropic materials by Wasscher [2] who solved the case of circular or rectangular samples. The main result in this extention is the determination of the main ratio of the in plane resistivity anisotropy. In this paper we shall examine the possibility to determine the anisotropy ratio in the case of circular samples, comparing experimental values treated by Wasscher's empirical method and theoretical ones, that are obtained by treating the data using an appropriate programme in the electronic computer.

2. PRINCIPLES OF THE WASSCHER'S METHOD [2]

Suppose that we have a flat circular anisotropic sample (Fig. 1) of radius r and thickness d. The two principal directions of resistivity x_1 and x_2 are in the plane of the sample, while the third one x_3 is normal to it.

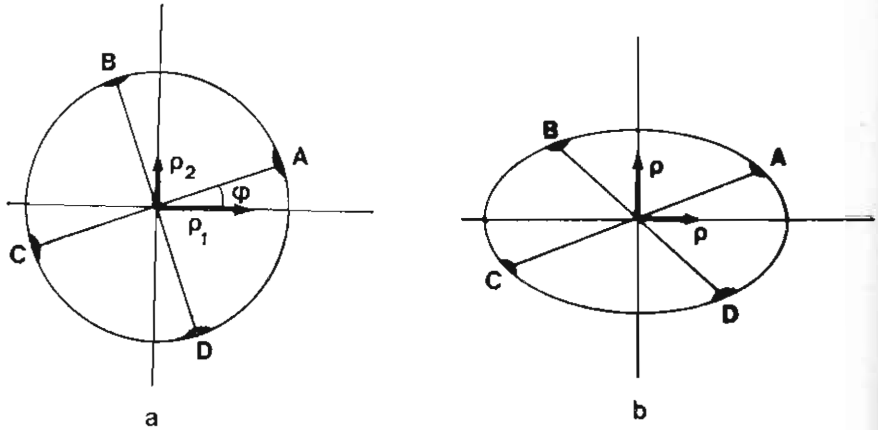

Fig. 1 (a) A circular anisotropic sample with four contacts. (b) The equivalent elliptic isotropic sample.

According to the Wasscher's proof this anisotropic sample is electrically equivalent to an isotropic one with elliptic shape (Fig. 1) and the following characteristics

a. thickness $d' = d(\rho_3/\rho)^{1/2}$ (1)

b. semi-axes $a = r(\rho_1/\rho)^{1/2}$ (2)

$b = r(\rho_2/\rho)^{1/2}$ (3)

c. resistivity $\rho = (\rho_1\rho_2\rho_3)^{1/3}$. (4)

Consider now four contacts A, B, C, D formed along the circumference of the circular sample on two perpendicular diameters (Fig. 1). Then we may define the "resistances"

$$R_1 = \frac{V_D - V_C}{I_{AB}}, \quad R_2 = \frac{V_A - V_D}{I_{BC}} \quad \text{and} \quad R_{12} = \frac{V_A - V_C}{I_{BD}} \tag{5}$$

As Van der Pauw proved [1] the resistivity ρ is given by the following relation

$$\rho = \frac{\pi d'}{2\ln 2}(R_1 + R_2) f\left(\frac{R_1}{R_2}\right) \tag{6}$$

The corrective factor $f(R_1/R_2)$ is a function of the R_1/R_2 ratio and its values are given graphically [1] or by the parametric expression [2]

$$f\left(\frac{R_1}{R_2}\right) = \frac{\log \frac{1}{4}}{\log \frac{1}{2}(1+x) + \log \frac{1}{2}(1-x)} \ ,$$

$$\frac{R_1}{R_2} = \frac{\log \frac{1}{2}(1-x)}{\log \frac{1}{2}(1+x)} \tag{7}$$

for $-1 < x < 1$.

Combining eqs. (1), (4) and (6) the following relation is obtained

$$(\rho_1\rho_2)^{1/2} = \frac{\pi d}{2\ln 2}(R_1 + R_2) f\left(\frac{R_1}{R_2}\right) \tag{8}$$

By conformal mapping of the ellipse into a unit circle, Wasscher obtained the following expressions for the resistances R_1 and R_2:

$$R_1 = \frac{(\rho_1\rho_2)^{1/2}}{\pi d} \ln\left[\frac{2}{1 - k\,sn(2u)}\right] \tag{9}$$

$$R_2 = \frac{(\rho_1\rho_2)^{1/2}}{\pi d} \ln\left[\frac{2}{1 + k\,sn(2u)}\right] \tag{10}$$

where $sn(2u) = sn\left[4K(m)\frac{\varphi}{\pi}\right]$ is an elliptical function of the complete elliptical integral of the first kind $K(m)$, φ being the angle between one of the principal axes of resistivity and a line connecting two opposite contacts. The modulus k is related to the parameter m by the expression

$$k = \sqrt{m} \tag{11}$$

The maximum value of the ratio R_1/R_2 occurs for $\varphi = \pi/4$ where $sn(2u) = 1$.

Combining eqs. (9) and (10) we have, for the extremum values of the resistances R_1, R_2, the following expressions

$$(R_1)_{max} = \frac{(\rho_1\rho_2)^{1/2}}{\pi d} \ln \frac{2}{1 - k} \qquad (12)$$

$$(R_2)_{min} = \frac{(\rho_1\rho_2)^{1/2}}{\pi d} \ln \frac{2}{1 + k} \qquad (13)$$

$$\left(\frac{R_1}{R_2}\right)_{max} = \frac{\ln\left[\frac{1}{2}(1 - k)\right]}{\ln\left[\frac{1}{2}(1 + k)\right]} \qquad (14)$$

From eq. (14) the value of the modulus k is determined.

Wasscher expressed graphically the dependence of the normalized ratio R_1/R_2 on the angle φ (Fig. 2) for four values of the anisotropy ratio $\lambda = \rho_1/\rho_2$ $(\rho_1 > \rho_2)$. So the main problem is the reproduction of the group curves of Fig. 2 or in other words finding of the exact value of the ratio λ which corresponds to the experimental value $(R_1/R_2)_{max}$ and the correspondence between the ratio R_1/R_2 and the angle φ for the above concrete value λ. Valassiades and Economou [3] gave a solution to the problem using an empirical relation which reproduces the curves of Fig. 2.

3. CORRELATION OF THE RATIOS λ AND (R_1/R_2)

For a definitive value λ it is possible to calculate the (g)nome Q(k) from the relation [2].

$$Q(k) = \exp\left[-\frac{\pi K(1 - m)}{K(m)}\right] = \frac{(\lambda^{1/2} - 1)^4}{(\lambda - 1)^2} \qquad (15)$$

From the above relation we may also find the parameter m and the modulus k. The value of the integral K(m) for the calculations, is found by the approximate formula (C. Hastings [4])

$$K(m) = [A_0 + A_1(1 - m) + A_2(1 - m)^2 + A_3(1 - m)^3 + A_4(1 - m)^4]$$

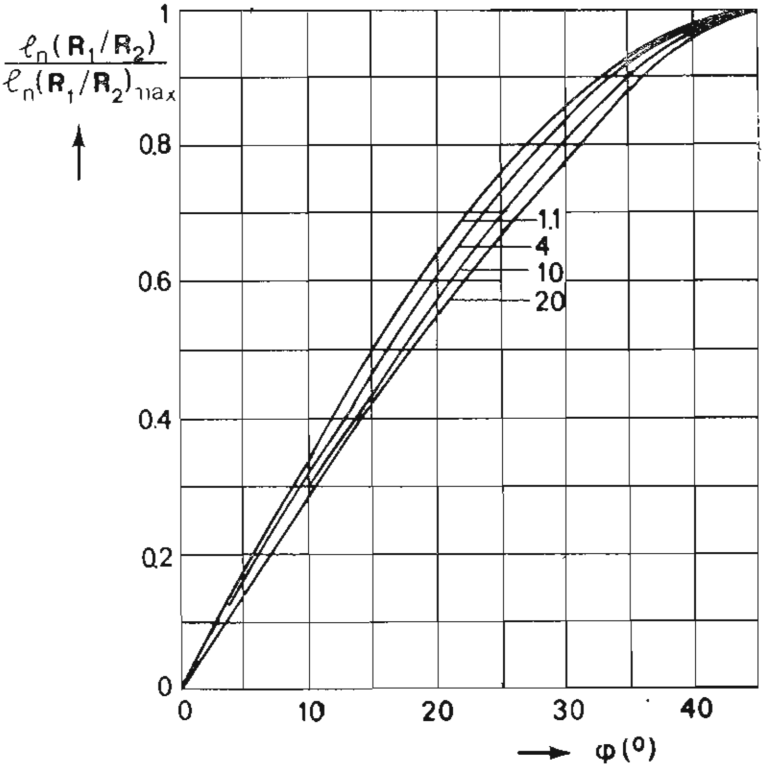

Fig. 2 The dependence of $ln(R_1/R_2)$ on the angle of rotation φ for four values of the anisotropy ratio λ [2].

$$+ [B_0 + B_1(1 - m) + B_2(1 - m)^2 + B_3(1 - m)^3 + B_4(1 - m)^4]$$

$$\ln \frac{1}{1 - m} + \mathscr{E}(m) \qquad (16)$$

where A and B are numerical constants and $|\mathscr{E}(m)| \leqslant 2 \times 10^{-8}$. The exact value of K(m) is found by the relation

$$K(m) = \frac{\pi}{2} + 2\pi \sum_{s=1}^{\infty} \frac{Q^s}{1 + Q^{2s}} \qquad (17)$$

while the value of the elliptical function sn(2u) is calculated using the relation

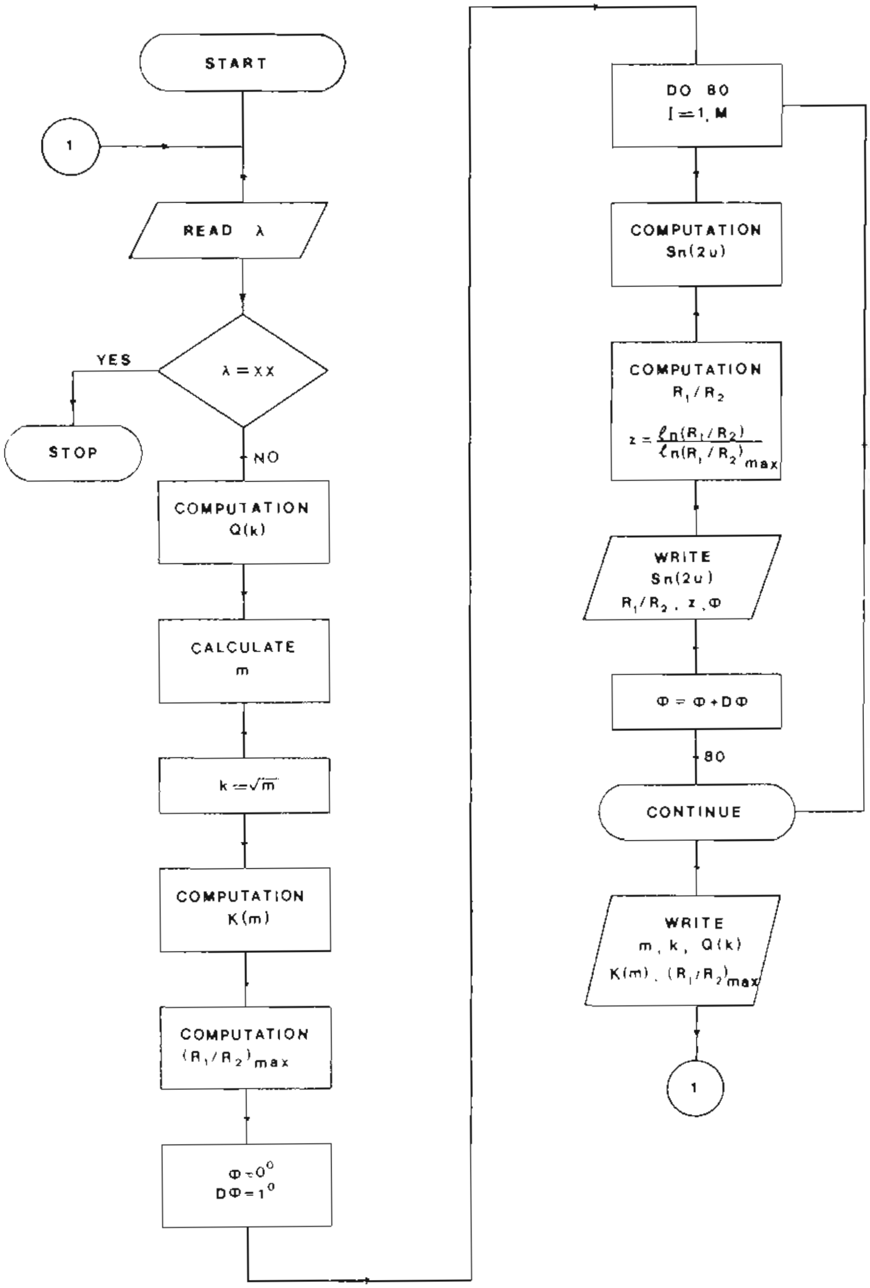

Fig. 3 Flow chart diagram for the reproduction of Wasscher's curves.

$$\mathrm{sn}(2u) = \frac{2\pi}{m^{1/2}K(m)} \sum_{n=0}^{\infty} \frac{Q^{n+1/2}}{1 - Q^{2n+1}} \sin[(2n+1)v] \tag{18}$$

where $v = \dfrac{\pi 2u}{2K(m)}$.

Using the programme presented by the flow chart of Fig. 3, we obtain the values for m, k, Q(k), K(m), $(R_1/R_2)_{max}$, $(R_1/R_2)_\varphi$, sn(2u), φ.

Therefore it is possible to construct a diagram (Fig. 4) showing the dependence of the ratio R_1/R_2 on the angle φ for any value λ without any restriction.

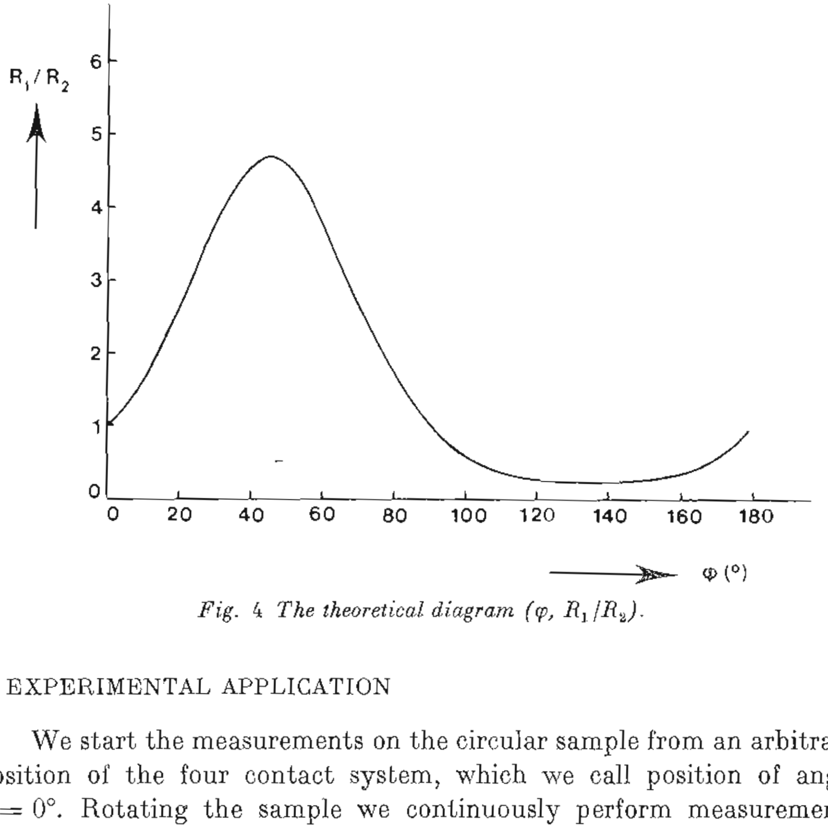

Fig. 4 The theoretical diagram (φ, R_1/R_2).

4. EXPERIMENTAL APPLICATION

We start the measurements on the circular sample from an arbitrary position of the four contact system, which we call position of angle $\theta = 0°$. Rotating the sample we continuously perform measurements for several values of the arbitrary angle θ. After that we construct a diagram of the experimental values R_1/R_2 vs the angle θ, using the same scale with the one used for the diagramm of the theoretical values (φ, R_1/R_2).

After that we allow the experimental diagram to slid on the theoretical ones until we succeed to fit it with someone of them. Thus the exact value of λ is determined and the relation between the angles θ and φ is also obtained (Fig. 5).

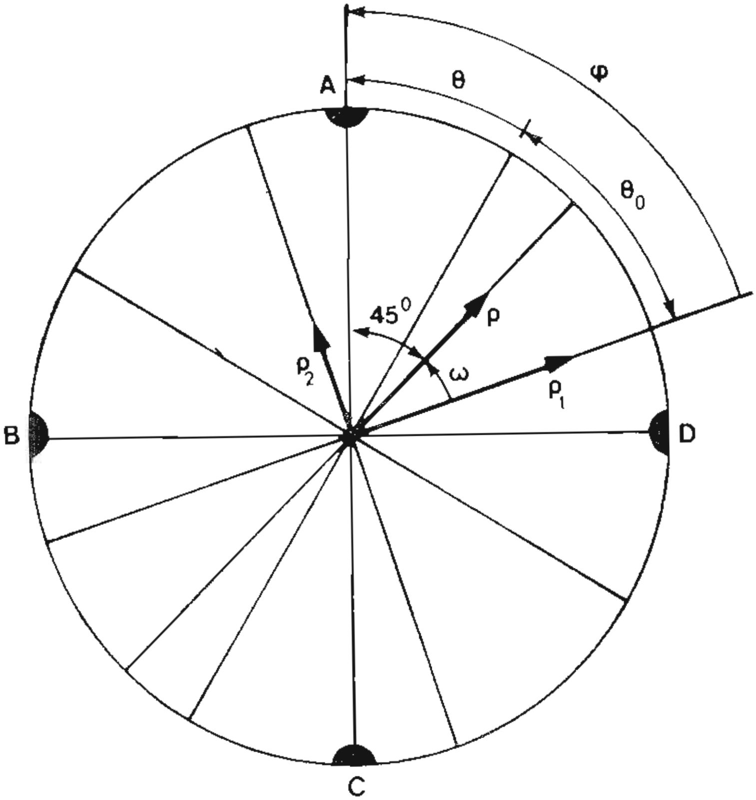

Fig. 5 Correlation of the angles φ, θ and ω.

If we have not measured experimentally the value $(R_1)_{max}$, we can find it as follows. Let R_1 be the experimental value which corresponds to the angle $\theta = \varphi - \theta_0$ (Fig. 5). Then from eqs. (9) and (12) we have

$$(R_1)_{max} = R_1 \frac{\ln\left[\frac{2}{1-k}\right]}{\ln\left[\frac{2}{1-k\,sn(2u)}\right]} \qquad (19)$$

In a similar way we obtain

$$(R_2)_{min} = R_2 \frac{\ln\left[\frac{2}{1+k}\right]}{\ln\left[\frac{2}{1+k\,sn(2u)}\right]} \qquad (20)$$

Knowing the values k, $(R_1)_{max}$ and $(R_2)_{min}$ we can calculate, using eq. (12) or (13), the product $\rho_1\rho_2$. Finally, for the main in plane resistivities ρ_1 and ρ_2, we have the following expressions:

$$\rho_1 = \frac{\lambda^{1/2}\pi d(R_1)_{max}}{\ln\frac{2}{1-k}} = \frac{\lambda^{1/2}\pi d(R_2)_{min}}{\ln\frac{2}{1+k}} \qquad (21)$$

$$\rho_2 = \frac{\lambda^{-1/2}\pi d(R_1)_{max}}{\ln\frac{2}{1-k}} = \frac{\lambda^{-1/2}\pi d(R_2)_{min}}{\ln\frac{2}{1+k}} \qquad (22)$$

5. THE DEPENDENCE OF THE RESISTIVITY ON THE ANGLE φ

The values of the resistivities ρ_1 and ρ_2, which are determined by eqs. (21) and (22), are the maximum and the minimum values respectively. The resistivity along an arbitrary direction in the sample's plane is given by the well known relation [5]

$$\rho = \rho_1\cos^2\omega + \rho_2\sin^2\omega \qquad (23)$$

where ω is the angle between the direction of ρ with the direction of ρ_1. The relation between ω and φ is (Fig. 5)

$$\omega = \varphi - 45^{\circ} \qquad (24)$$

Combining eqs. (21), (22) and (23) we obtain the equation

$$\rho = \frac{\pi d(R_1)_{max}}{\ln\frac{2}{1-k}}(\lambda^{1/2}\cos^2\omega + \lambda^{-1/2}\sin^2\omega) \qquad (25)$$

In Fig. 6 the polar plot of the resistivity ρ (eq. (23) or (25)) is given. It

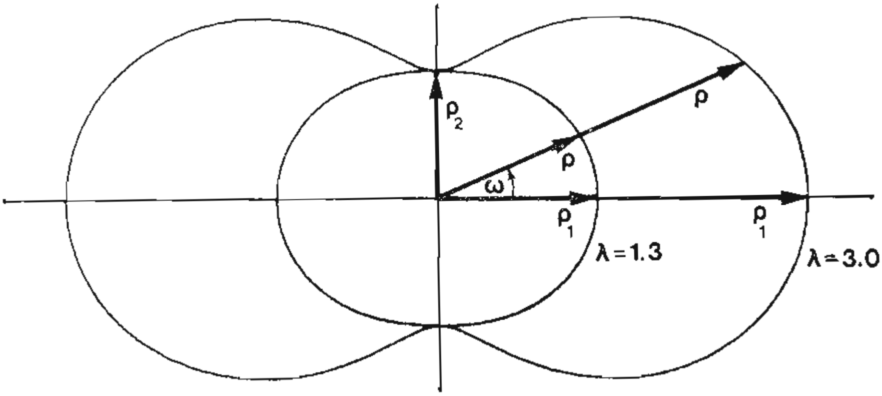

Fig. 6 Polar plot of the resistivity ϱ for two values of the ratio λ.

is evident that for two mutual perpendicular directions the anisotropy ratio is

$$\lambda_\omega = \frac{\lambda\cos^2\omega + \sin^2\omega}{\lambda\sin^2\omega + \cos^2\omega} \tag{26}$$

From eq. (25) it also follows that the sum of resistivities in the two perpendicular directions is constant and equal to the mean resistivity of the sample.

$$<\rho> = \frac{\rho_1 + \rho_2}{2} = \frac{\rho_\omega + \rho_{\omega+90^\circ}}{2} = \text{const} \ . \tag{27}$$

This value differs from that one (ρ_V) which we obtain by using eq. (8) if we consider the material as isotropic. So, when we make measurements for $\varphi = 0^\circ$ or $\varphi = 90^\circ$ ($\omega = \pm 45^\circ$) then $R_1 = R_2 = R$, $f(R_1/R_2) = 1$ and

$$\rho_V = \frac{\pi d}{\ln 2} R \tag{28}$$

which is not the actual mean value $<\rho>$.

From eq. (19) for $\varphi = 0$, $\text{sn}(2u) = 0$ we have

$$\frac{(R_1)_{max}}{\ln \frac{2}{1-k}} = \frac{R}{\ln 2} \tag{29}$$

and thus the eqs. (21) and (22) become

$$\rho_1 = \frac{\lambda^{1/2}\pi d R}{\ln 2} = \lambda^{1/2}\rho_V \qquad (30)$$

$$\rho_2 = \frac{\lambda^{-1/2}\pi d R}{\ln 2} = \lambda^{-1/2}\rho_V \qquad (31)$$

so that

$$<\rho> = \frac{\rho_V}{2}(\lambda^{1/2} + \lambda^{-1/2}) \ . \qquad (32)$$

Therefore $<\rho>$ is proportional to ρ_V while the proportionality constant depends on the main anisotropy ratio λ.

6. THE DETERMINATION OF THE MAGNETORESISTANCE AND OF THE HALL COEFFICIENT

The application upon the material of a magnetic field B results in introducing an anisotropy in the electrical resistivity with principal directions in general different from those of crystallographic axes which usually define the principal directions of the zero-field resistivity [6, 7]. This phenomenon, known as magnetoresistance skewness [8, 9], is directly connected with the weak-field magnetoresistance coefficients ρ_{ijkl}. Thus the measurement of the resistivity in the presence of the magnetic field leads to the determination of the ρ_{ijkl} coefficients. As Wasscher [2] has indicated, the previous analysis of the anisotropic zero-field resistivity may also be applied for the determination of the resistivity under the influence of the magnetic field. The needed configurations depend on the crystal class of the material [6] and the orientation of the surface layers or the epitaxial films [7].

In the simple case where the application of the magnetic field B does not change the main directions of resistivity, the magnetoresistance may be calculated from the following relations

$$\frac{\Delta\rho_1}{\rho_1} = \left(\frac{\lambda'}{\lambda}\right)^{1/2} \frac{(R_1')_{max}}{(R_1)_{max}} \frac{\ln \dfrac{2}{1-k}}{\ln \dfrac{2}{1-k'}} - 1 \qquad (33)$$

$$\frac{\Delta\rho_2}{\rho_2} = \left(\frac{\lambda}{\lambda'}\right)^{1/2} \frac{(R_2')_{min}}{(R_2)_{min}} \frac{\ln \dfrac{2}{1+k}}{\ln \dfrac{2}{1+k'}} - 1 \tag{34}$$

where the prime quantities are the ones in the presence of the magnetic field.

In an arbitrary direction the magnetoresistance is

$$\left(\frac{\Delta\rho}{\rho}\right) = \frac{\rho_1\rho_2\left[\left(\dfrac{\Delta\rho_1}{\rho_1}\right)\dfrac{\cos^2\omega}{\rho_2} + \left(\dfrac{\Delta\rho_2}{\rho_2}\right)\dfrac{\sin^2\omega}{\rho_1}\right]}{\rho} \tag{35}$$

Finally the Hall coefficient is calculated by the Van der Pauw relation [1]

$$R_H = \frac{d}{B}\,\Delta R_{12} \tag{36}$$

where the magnetic field B is perpendicularly applied on the sample and ΔR_{12} is the change in the resistance R_{12} (eq. (5)) due to the presence of the field.

REFERENCES

1. L. J. van der Pauw, Philips Res. Rep. **13**, 1 (1958).
2. J. D. Wasscheh, Philips Res. Rep. Suppl. **8**, 1 (1969).
3. O. Valassiades and N. A. Economou, Sci. Annals. Fac. Phys. and Mathem., Univ. of Thessaloniki, **17**, 323 (1977).
4. Handbook of Mathematical Functions, Dover Publ., N. York 1970.
5. J. F. Nye, Physical Properties of Crystals, Oxford Univ. Press, London 1967.
6. D. S. Kyriakos and N. A. Economou, Phys. Stat. Sol. (b) **94**, 549 (1979).
7. D. S. Kyriakos, N. A. Economou and R. S. Allgaier, Revue Phys. Appl. **15**, 733 (1980)
8. R. S. Allgaier, J. B. Restorff and B. Houston, Appl. Phys. Lett. **34**, 158 (1979).
9. D. S. Kyriakos and N. A. Economou, Appl. Phys. Lett., **35**, 894, (1979).

ΠΕΡΙΛΗΨΗ

ΜΙΑ ΕΠΕΚΤΑΣΗ ΤΗΣ ΜΕΘΟΔΟΥ WASSCHER ΓΙΑ ΓΑΛΒΑΝΟΜΑΓΝΗΤΙΚΕΣ ΜΕΤΡΗΣΕΙΣ ΣΕ ΑΝΙΣΟΤΡΟΠΑ ΥΛΙΚΑ

ὑπὸ

Δ. Σ. ΚΥΡΙΑΚΟΥ καὶ Ν. Α. ΟΙΚΟΝΟΜΟΥ

(Ἐργαστήριο Β' Ἕδρας Φυσικῆς Παν/μίου Θεσ/νίκης)

Ἕνα κατάλληλο πρόγραμμα ὑπολογιστῆ χρησιμοποιεῖται γιὰ νὰ ἁπλοποιήσει τοὺς ὑπολογισμοὺς στὴ μέθοδο τοῦ Wasscher γιὰ γαλβανομαγνητικὲς μετρήσεις σὲ ἀνισότροπα ὑλικά. Ἔτσι γιὰ ἐπίπεδα κυκλικὰ δείγματα προσδιορίζεται ὁ λόγος τῆς κύριας ἀνισοτροπίας τῶν εἰδικῶν ἀντιστάσεων πάνω στὸ ἐπίπεδο τοῦ δείγματος καὶ ἐξάγονται πρόσφοροι τύποι γιὰ τὸν ὑπολογισμὸ τῶν κυρίων εἰδικῶν ἀντιστάσεων.