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A GENERALIZATION OF KOLMOGOROV'S INEQUALITY

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Abstract: *A generalization of Kolmogorov's inequality is presented. If the random variables form a martingale difference sequence and have moments of order r, $1<r\leqslant 2$, then an inequality similar to that of Hajek-Renyi is proved. Examples demonstrating the applicability of our results to the convergence of random variables are also given.*

1. INTRODUCTION

The well known inequality of Kolmogorov, for independent random variables having second moments, was generalized by J. Hajek and A. Renyi [2]. It turns out, that the same proof carries through, if the random variables form a martingale difference sequence which includes the independence case. In the present paper we extend their inequality for a martingale difference sequence with finite moments of order r, $1<r\leq 2$.

The interesting part is that we do not need to have finite variances but finite moments of order r, $1<r\leq 2$.

For the proof we use an inequality which is due to Esseen and Von Bahr [1]. This replaces Minkowski's inequality which was used in [3] for a different problem.

Our main result is summarized in the following:

THEOREM. Let $X_1, X_2,\ldots,X_n$ be a sequence of random variables such that:

i) $E|X_i|^p < \infty$ for $i=1, 2, \ldots, n$, $1<p\leq 2$.

ii) $E(X_i|X_1,\ldots,X_{i-1})=0$ for $i=2,\ldots,n$

If $c_1, c_2, \ldots, c_n$ is a non-increasing sequence of positive constants, then for any positive integers m, n with $m<n$ and arbitrary $h>0$,

$$(1)\quad P\left(\max_{m\leqslant k\leqslant n} c_k|X_1+\ldots+X_k|\geq h\right)\leq$$

$$\leq\alpha\left(c_m^p\sum_{i=1}^{m}E|X_i|^p+\sum_{k=m+1}^{n}c_k^p E|X_k|^p\right)/h^p$$

where $1<\alpha=f(p)\leq 2^{2-p}<2$ for $1<p<2$ and $f(2)=1$.

PROOF. Let $S_i=X_1+\ldots+X_i$, $A_i(i=m,\ldots,n)$ be the event $(c_m|S_m|<h,\ldots,c_{i-1}|S_{i-1}|<h,\ c_i|S_i|\geq h)$, then $A_i\cap A_j=0$ for $i\neq j$ and

$$A=\bigcup_{i=m}^{n}A_i,\ \text{where}\ A=\{\max_{m\leqslant i\leqslant n}c_i|S_i|\geq h\}$$

Now consider the random variable

$$(2)\ Z=c_m^p|S_m|^p+\sum_{k=m+1}^{n}c_k^p(|S_k|^p-|S_{k-1}|^p)\ (1-I_{k-1}-\ldots-I_m)$$

where I_k is the indicator random variable of the event A_k, then it is easy to see that $Z\geq 0$ everywhere and $Z\geq h^p$ in A. Hence if $F(x_1,\ldots,x_n)$ is the joint distribution of $(X_1,\ldots,X_n)=X$ we have

$$(3)\quad P(\max_{m\leqslant i\leqslant n}c_i|S_i|\geq h)=P(X\in A)=\int_A dF\leq\left(\int_A ZdF\right)/h^p\leq(EZ)/h^p$$

Applying now Essseen-Von Bahr's inequality, see (1) i.e.,

$$(4)\qquad |S_k|^p=|S_{k-1}+X_k|^p\leq|S_{k-1}|^p+\alpha|X_k|^p+p|S_{k-1}|^{p-1}(\text{sign. }S_{k-1})X_k$$

obtain from (2)

$$(5)\quad Z\leq c_m^p|S_m|^p+\sum_{k=m+1}^{n}c_k^p\left(\alpha|X_k|^p+p|S_{k-1}|(\text{sign }S_{k-1})X_k\right)(1-I_{k-1}-\ldots-I_m)$$

Thus taking expectations and observing that

$$0\leq 1-I_{k-1}-\ldots-I_m\leq 1\ \text{and}\ E|S_m|^p\leq\alpha\sum_{i=1}^{m}E|X_i|^p$$

we establich the desired result Q.E.D.

From (5) it is clear that the same inequality is true if instead of $E(X_i|X_1,...,X_{i-1})=0$, $i=2,...,n$ we assume that $(\text{sign } S_{i-1})\ E(X_i|X_1,...,X_{i-1})\leq 0$, $i=2,....,n$.

The value $p=2$ gives Hajek-Renyi's inequality.

2. APPLICATIONS

Let the infinite sequence of randon variables $X_1,...$, then,

COROLLARY 1. If i) $b_i \uparrow \infty$

ii) $E(X_i|X_1,...,X_{i-1})=0 \qquad i=2,3,...$

iii) $\sum_{i=1}^{\infty}(E|X_i|^p / b_i^p) < \infty$ for some $1<p\leq 2$.

then $\lim_{n\to\infty}\left(\sum_{i=1}^{n}X_i\right)/b_n=0$, a.s. and in L^p.

PROOF. Take $c_i=(1/b_i)\downarrow 0$ and apply inequality (1), then from

$$\lim_{m\to\infty}\sum_{i=m+1}^{\infty}((E|X_i|^p)/b_i^p=0$$

as the tail of a converging series, obtain

$\lim_{m\to\infty}\left(\sum_{i=1}^{m}E|X_i|^p\right)/b_m^p=0$ by Kronecker's lemma, which proves the L^p - convergence, furthermore

$$\lim_{m\to\infty}P(\max_{m\leqslant n}\left(|X_1+...+X_n|/b_n\right)\geq h)=0 \qquad \text{Q.E.D.}$$

EXAMPLE. If $E|X_i|^p<c<\infty$ for $1<p\leq 2$, $i=1, 2, ...$,

and $E(X_i|X_1,..., X_{i-1})=0 \quad i=2,...$

then for any $0<q<p$.

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}X_i\right)/n^{1/q}=0 \text{ a.s. and in } L^p.$$

PROOE. Take $b_n = n^{1/q}$, then $\sum_{i=1}^{\infty} (E|X_i|^p)/i^{p/q} \leq c \sum_{i=1}^{\infty} i^{-p/q} < \infty$

COROLLARY 2. If i) $b_i \uparrow \infty$ ii) X_i is any sequence of random variables such that

$$\sum_{i=1}^{\infty} (E|Y_i|^p / b_i^p) < \infty \quad \text{where } Y_i = X_i - E(X_i|X_1, \ldots, X_{i-1})$$

$i = 2, 3, \ldots, \quad Y_1 = X_1,$

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} Y_i \right) / b_n = 0 \text{ a.s. and in } L^p.$$

PROOF. The sequence Y_i satisfies the requirements of the previous Corollary.

Of course if $0 < p \leq 1$, Corolarry 1 is true for an arbitrary sequence of random variables, as it is proved in [3].

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ΠΕΡΙΛΗΨΗ

ΜΙΑ ΓΕΝΙΚΕΥΣΗ ΤΗΣ ΑΝΙΣΟΤΗΤΑΣ ΤΟΥ KOLMOGOROV

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Δίνεται μία γενίκευση τῆς ἀνισότητας τοῦ Kolmogorov γιὰ τυχαῖες μεταβλητὲς πού ἔχουν ροπὲς τάξης r, $1 < r \leq 2$. Οἱ τυχαῖες μεταβλητὲς δὲν εἶναι ἀπαραίτητο νὰ εἶναι ἀνεξάρτητες ἀλλὰ ἀρκεῖ νὰ ἱκανοποιοῦν τὴ σχέση

$$E(X_i \mid X_1, \ldots, X_{i-1}) = 0.$$

Ἐπίσης δίνονται ἐφαρμογὲς τῆς ἀνισότητας γιὰ νὰ διαπιστωθεῖ ἡ ἰσχυρὴ σύγκλιση μιᾶς ἀκολουθίας τυχαίων μεταβλητῶν.