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RESTRICTIVE STABILITY OF DIFFERENCE EQUATIONS

by

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Abstract: *Two linear, difference equations are called t_∞- similar if the series of general term $\| S(t)B(t) - A(t)S(t\text{-}1) \|$, where $A(t)$, $B(t)$ are the mappimgs of the left sides of the corresponding difference equations and $S(t)$ a linear homeomorphism, is convergent. A linear difference equation is restrictively stable if both the fundamental solution of the equation and its inverse are bounded. The main result of this paper is the proposition: «A linear difference equation is restrictively stable if and only if the linear mapping of its left side is t_∞ - similar to the identity mapping».*

1. Difference equations are often used to analyze the so-called sampled — data systems, in which the stability problems are considered to be important [8]. It seems, however, that if we are concerned with the stability of difference equations, not so many papers have been appeared so far. The purpose of this note is to introduce the concept of restrictive stability of difference equations and to study the invariance of this under some types of similarities devised by *L. Markus*[7]. Related work for differential equations has been done by *G. Askoli*, *G. Sansone* and *R. Conti* [1, 4, 5].

Firstly we introduce some notation. Let $I = \{0,1,2,....,\}$; $I_{t_0} = \{t_0, t_0+1,...,\}$, $t_0 \in I$; E a Banach space with norm $|\cdot|$; L(E) the linear space of linear continuous mappings of E with norm $\|\cdot\|$ induced by $|\cdot|$; L_h (E) the space of linear homeomorphisms of E ; M The identity mapping of E and finally 0 the zero mapping of E. In the following juxtaposition of mappings means composition of mappings.

2. Let $A: I_1 \rightarrow L_h(E) : t \rightarrow A(t)$ be a given mapping and X(t) be the principal fundamental solution of the difference equation

$$x(t) = A(t)x(t-1) \quad , \qquad t \in I_1, \tag{1}$$

Definition 1. We say that (1) is restrictively stable if there exist positive constants M_1 and M_2, such that

$$\| X(t) \| < M_1 \quad , \quad \| X^{-1}(t) \| < M_2 \quad , \quad t \in I \ .$$

Definition 2. Let $A : I_1 \to L(E)$ and C a constant mapping. We say that A is reducible to the constant mapping C (resp. C is reducible to A) if there exists a mapping $S : I \to L_h(E)$ such that

$$C = S^{-1}(t)A(t)S(t-1) \ [\text{resp. } A(t) = S^{-1}(t)CS(t-1)], \qquad t \in I_1 \, .$$

In that case the substitution

$$x(t) = S(t)y(t) \quad (2) \quad , \quad [\text{resp. } y(t) = S(t)x(t)] \qquad (2')$$

transforms (1) to

$$y(t) = Cy(t-1), \qquad (3)$$

[resp. the substitution (2′) transforms (3) to (1)]. If C = M, we say that A is reducible to the identity.

Example 1. [6]. Let N be a constant positive integer and consider the periodic difference equation

$$x(t) = A(t)x(t-1) \, ; \ [A(t+N) = A(t)], \qquad t \in I_1 \qquad (4)$$

where $x \in R^n$. Then, it is well known that every principal solution X(t) of (4) has the form

$$X(t) = P(t)Q^{t/N} \ ; \ [P(t+N) = P(t)] \ , \qquad t \in I_1$$

where P(t) is a nonsingular matrix and Q a nonsingular constant matrix. The substitution

$$x(t) = P(t)y(t)$$

transforms (4) to

$$y(t) = Q^{1/N} \ y(t-1) \ .$$

Therefore a periodic matrix is always reducible to a constant matrix. As for the case of differential equations [[3] p. 201] the above result can be generalized to an infinite dimensional Banach space.

Example 2. Let the difference equation

$$x(t) = A(t)x(t-1) = (1+2^{-t})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t-1), \qquad t \in I_1, \tag{5}$$

Then, A(t) is reducible to the identity. In fact, the substitution

$$x(t) = S(t)y(t) = \left[\prod_{s=1}^{t} (1 + 2^{-s})\right]\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{t} y(t)$$

transforms (5) to

$$y(t) = y(t-1)$$

and obviously, $S : I \to L_h(R^2)$.

Definition 3. Two mappings $A : I \to L(E)$, $B : I \to L(E)$ are called t — similar if there exists a mapping $S : I_1 \to L_h(E)$ such that

$$B(t) = S^{-1}(t)A(t)S(t-1) \quad \text{or} \quad A(t) = S^{-1}(t)B(t)S(t-1).$$

This also means that, by the substitution (2), equation (1) is transformed to

$$y(t) = B(t)y(t-1) \tag{6}$$

(resp. by the substitution (2′), equation (6) is transformed to (1)).

The t - similarity is an equivalence relation and it coincides with reducibility, if $B(t) = C$.

Definition 4. Two mappings $A : I_1 \to L(E)$, $B : I_1 \to L(E)$ are called t_∞ — similar if there exist a mapping $S : I \to L_h(E)$ and a mapping $F : I_1 \to L(E)$ satisfying

$$\sum_{s=1}^{\infty} \| F(s) \| < \infty \ , \tag{7}$$

such that

$$S(t)B(t) - A(t)S(t-1) = F(t). \tag{8}$$

The t_∞ — similarity is an equivalence relation and if $F(t) \equiv 0$, t_∞ — similarity becomes t — similarity. Moreover, if $S(t) = M$, condition (7) is equivalent to the convergence of the series

$$\sum_{t=1}^{\infty} \| A(t) - B(t) \| .$$

Theorem 1. The difference equation (1) is restrictively stable if and only if, $A: I_1 \to L_h(E)$ is reducible to the identity.

Proof. Let A(t) be reducible to the identity. According to Definition 2, if S(t) is the mapping in (2) and X(t) the principal funtamental solution of (1)

$$\| X(t) \| \leqslant \| S(t)C \| = M_1 \quad , \quad \| X(t) \| \leqslant \| S^{-1}(t)C \| = M_2 \,, \quad t \in I.$$

Conversely, if (1) is restrictive stable, then, because

$$X(t) - A(t)X(t-1) = 0 ,$$

(8) is satisfied with $S(t) = X(t)$, $B(t) = M$, $F(t) = 0$, for all $t \in I$. So A(t) is reducible to the identity.

Remark 1. From Theorem 1, it follows that the class of reducible to the identity mapings coincides with the class of mappings which correspond to linear restrictively stable differential equations.

Remark 2. Consider the difference equation

$$z(t) = [A^{-1}(t)]^* \; z(t-1) , \tag{9}$$

where $z \in E$, E a Hilbert space, $[A^{-1}(t)]^*$ is the adjoint operator of $A^{-1}(t)$ [[3] p. 31]. Equation (9) is the so - called adjoint equation of (1). The principal fundamental solution Z(t) of (9) equals to $[X^{-1}(t)]^*$. Hence, [[8] p. 214].

$$\| Z(t) \| = \| [X^{-1}(t)]^* \| = \| X^{-1}(t) \| .$$

Therefore, according to Definition 1, restrictive stability of (1) is equivalent to the stability of equation (1) and its adjoint equation (9) [3].

By virtue of Theorem 1, if (1) and its adjoint equation are both stable, A(t) is reducible to the identity.

Theorem 2. The difference equation (6) is restrictively stable if and only if M and $B : I \to L_h(E)$ are t_∞ — similar.

Proof. Let M and B(t) are t_∞ — similar. Then (8) holds with A(t) = M. Consider the operator equation

$$S(t) = [S(t_0)Y(t_0) + \sum_{t_0+1}^{t} F(s)Y(s-1)]Y^{-1}(t), \qquad t \in I_{t_0}, \quad (10)$$

where Y(t) is the principal fundamental solution of (6). Observe that S(t) given by (10) is a solution of

$$S(t)B(t) - S(t-1) = F(t) \quad , \quad t \in I, \qquad (11)$$

From (10), we have

$$Y(t)Y^{-1}(t_0) = S^{-1}(t)[S(t_0) + \sum_{s=t_0+1}^{t} F(s)Y(s-1)Y^{-1}(t_0)],$$

and therefore

$$\| Y(t)Y^{-1}(t_0) \| \leqslant c_1 + \sum_{s=t_0+1}^{t} c_2 \| F(s) \| \; \| Y(s-1)Y^{-1}(t_0) \| \leqslant$$

$$\leqslant c_1 + \sum_{s=t_0}^{t-1} c_2 \| F(s+1) \| \; \| Y(s)Y^{-1}(t_0) \| .$$

Applying Gronwall's inequality [2], for all $t \in I_{t_0}$,

$$\| Y(t)Y^{-1}(t_0) \| < c_1 \exp \sum_{s=t_0}^{t-1} c_2 \| F(s+1) \| < c_1 \exp \sum_{s=1}^{\infty} c_2 \| F(s) \| .$$

That is $\| Y(t)Y^{-1}(t_0) \|$ is bounded for all $t \in I_{t_0}$.

Moreover, in order to prove that $\| Y(t)Y^{-1}(t_0) \|$ is bounded for $t = 0, 1, \ldots, t_0$, we have to consider the backward solution of the operator equation

$$Y(t) = B(t)Y(t-1) ,$$

which is given by

$$Y(t) = B^{-1}(t+1)\,B^{-1}(t)\ldots B^{-1}(0)MY(t_0).$$

Using (10) and getting the backward solution of (11), we have

$$S(t) = [S(t_0)Y(t_0) - \sum_{s=t+1}^{t} F(s)Y(s-1)]Y^{-1}(t),$$

or

$$Y(t)Y^{-1}(t_0) = S^{-1}(t)[S(t_0) - \sum_{s=t}^{t_0-1} F(s+1)Y(s)Y^{-1}(t_0)] .$$

From this we get

$$\| Y(t)Y^{-1}(t_0) \| \leqslant c_1 + \sum_{s=t}^{t_0-1} c_2 \| F(s+1) \| \; \| Y(s)Y^{-1}(t_0) \|$$

and by Gronwall's inequality, again

$$\|Y(t)Y^{-1}(t_0)\| \leqslant c_1 \exp \sum_{s=t}^{t_0-1} c_2 \| F(s+1) \| \leqslant c_1 \exp \sum_{s=1}^{\infty} c_2 \| F(s) \| .$$

Conversely, if (6) is restrictively stable, then, because

$$Y(t) - B(t)Y(t-1) = 0 ,$$

(8) is satisfied with $A(t) = M$, $S(t) = Y(t)$ and $F(t) = 0$ and therefore M and $B(t)$ are t_∞ — similar.

Theorem 1 shows that if (1) is restrictively stable, then every t_∞ similar mapping $B : I \rightarrow L_h(E)$ of $A : I \rightarrow L_h(E)$ corresponds to an e-quation of the form (6) which is restrictively stable. This means that restrictive stability is an invariance property under t_∞ — similarity.

The following corollaries are concequences of Theorem 2.

Corollary 1. If (1) is restrictively stable and

$$\sum_{s=1}^{\infty} \| B(s) - A(s) \| ,$$

is a convergent series, then (6) is restrictively stable.

From Theorem 2 and Corollary 1 we have the following corollary.

Corollary 2. The difference equation (1) is restrictively stable if and only if the series

$$\sum_{s=1}^{\infty} \| A(s) - M \| ,$$

is convergent.

According to Remark 2, a necessary condition for the restrictive stability for an autonomous difference equation is given by the following theorem.

Theorem 3. A necessary condition for the restrictive stability of the difference equation

$$y(t) = Cy(t-1) \quad , \quad t \in I_1 ,$$

where $y \in E$, $C \neq O$ a constant mapping of E, is that the spectrum of C lies on the unit circle.

The condition of Theorem 1 is also sufficient if $E = R^n$ and all elementary divisors of the matrix C are linear.

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ΠΕΡΙΛΗΨΙΣ

ΠΕΡΙΟΡΙΣΤΙΚΗ ΕΥΣΤΑΘΕΙΑ ΤΩΝ ΔΙΑΦΟΡΩΝ ΕΞΙΣΩΣΕΩΝ

Ὑπὸ

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Σκοπὸς τῆς παρούσης ἐργασίας εἶναι νὰ εἰσάγῃ τὴν ἔννοιαν τῆς περιοριστικῆς εὐσταθείας εἰς τὰς γραμμικὰς διαφορῶν ἐξισώσεις καὶ νὰ δείξῃ ὅτι αὕτη μένει ἀναλλοίωτος δι' ὡρισμένων τύπων μετασχηματισμῶν ὁμοιότητος. Μία γραμμικὴ διαφορῶν ἐξίσωσις εἶναι περιοριστικῶς εὐσταθὴς ἐὰν ἡ ἐπιλύουσα αὐτῆς καὶ ἡ ἀντίστροφός της εἶναι ἀμφότεραι περατωμέναι. Δύο γραμμικαὶ διαφορῶν ἐξισώσεις εἶναι t_∞ — ὅμοιαι ἐὰν ἡ σειρὰ ἡ ἔχουσα γενικὸν ὅρον $\|S(t)B(t) - A(t)S(t\text{-}1)\|$, ἔνθα A(t), B(t) αἱ ἀπεικονίσεις ἀντιστοίχως τῶν ἀριστερῶν μελῶν τῶν ἐξισώσεων καὶ S(t) ὁ προαναφερθεὶς μετασχηματισμός, εἶναι συγκλίνουσα. Ἀποδεικνύεται ὅτι: «Μία γραμμικὴ διαφορῶν ἐξίσωσις εἶναι περιοριστικῶς εὐσταθὴς ἐὰν καὶ μόνον ἐὰν ἡ γραμμικὴ ἀπεικόνισις τοῦ ἀριστεροῦ της μέλους εἶναι t_∞ —ὁμοία τῆς ταυτοτικῆς ἀπεικονίσεως». Ἐκ τῆς προτάσεως ταύτης ἕπονται μερικὰ πορίσματα τὰ ὁποῖα δύνανται νὰ θεωρηθοῦν ὡς κριτήρια διὰ τὴν περιοριστικὴν εὐστάθειαν τῶν γραμμικῶν διαφορῶν ἐξισώσεων.