Sci. Annals, Fac. Phys. & Mathem., Univ. Thessaloniki, **14**, 75 (1974)

ON THE REPRESENTATION OF THE COSINE OF THE ANGLE OF REFRACTION ($cos\theta_2$) IN THE COMPLEX PLANE

By

J. SARAKINOS, J. SPYRIDELIS

(Laboratory of Physics (A), University of Thessaloniki)

(Received, 12.8.1974)

Abstract *The cosine of the angle of refraction ($cos\theta_2$) is investigated for all possible cases of propagation of light: from one dielectric into another, from a dielectric into a conductor, from one conductor into another conductor. In all these cases it is found that $cos\theta_2$ is represented by points lying in the fourth quadrant of the complex plane with limits corresponding to the positive part of the real axis and the negative part of the imaginary axis.*

INTRODUCTION

To investigate interference effects in thin films due to multiple reflections, the well known relation of the phase difference $\delta = \frac{2\pi n_2 d cos\theta_2}{\lambda}$ is used. In that relation the quantity $cos\theta_2$ is not always real and in most cases it is complex of the form $cos\theta_2 = c - id$, where c and d are real quantities. It is important to know whether these quantities are positive or negative especially in computer programmes dealing with ellipsometry problems.

In the following an attempt is made to investigate all the possible values of $cos\theta_2$ and to find the locus of the corresponding points in the complex plane.

THEORY

a. Let us consider first the propagation of a plane wave from one dielectric into another, both media assuming to be of infinite extent, the surface of contact being the plane $z = 0$. In this case the law of refraction is

$$n_1 sin\theta_1 = n_2 sin\theta_2 \quad (1)$$

or

$$\cos\theta_2 = \pm \left\{ 1 - \left(\frac{n_1}{n_2} \sin\theta_1 \right)^2 \right\}^{1/2}. \tag{2}$$

When $n_1 < n_2$ it is said that the second medium is optically denser than the first one. Therefore, using the inequality

$$A \equiv 1 - \left(\frac{n_1}{n_2} \sin\theta_1 \right)^2 > 0 \tag{3}$$

it follows that

$$\cos\theta_2 = \pm \sqrt{A} \tag{4}$$

In this case only the positive square root of A is taken into consideration [1]. Thus the position of $\cos\theta_2$ in the complex plane is represented by a point lying in the positive part of the real axis. (see Fig. 1, point (1)).

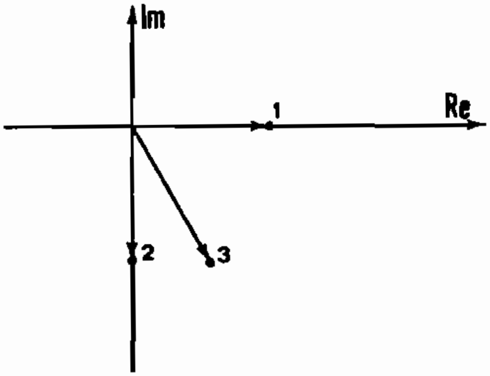

Fig. 1. Representation of $\cos\theta_2$ in the complex plane. Points (1),(2) and (3) represent all the possible situations of $\cos\theta_2$.

If however the second medium is optically less dense than the first one (i.e. if $n_1 > n_2$), one obtains a real value for $\cos\theta_2$, only for those incident angles θ_1 for which

$$\frac{n_1}{n_2} \sin\theta_1 \leqq 1 \tag{5}$$

The angle $\theta_1 = \theta_c$, for which $\sin\theta_c = \dfrac{n_2}{n_1}$, corresponds to the critical value of the angle of incidence. When $\theta_1 > \theta_c$ *total reflection* takes place.

In the latter case

$$\frac{n_1}{n_2} \sin\theta_1 > 1$$

or according to relation (2)

$$A = 1 - \left(\frac{n_1}{n_2} \sin\theta_1\right)^2 < 0 \tag{6}$$

and

$$\cos\theta_2 = \pm\, i \sqrt{|A|} \tag{7}$$

In this case we consider only the negative value of Eq. (7). The choise of the minus sign is dictated by the need to have a decreasing exponential dependence in the z direction for the transmitted wave [2]. Thus the position of $\cos\theta_2$ in the complex plane is represented by a point lying in the negative part of the imaginary axis. (see Fig. 1, point (2)).

b. Let us consider now the propagation of a plane wave from a dielectric into a conductor the surface of contact between them being the plane $z = 0$.

According to Snell's law,

$$n_1 \sin\theta_1 = \tilde{n}_2 \sin\theta_2 \tag{8}$$

or

$$\cos\theta_2 = \pm \left\{ 1 - \left(\frac{n_1 \sin\theta_1}{\tilde{n}_2}\right)^2 \right\}^{1/2} \tag{9}$$

Since $\tilde{n}_2$ is complex, so is $\cos\theta_2$ and therefore this quantity has no longer the simple significance of a refraction phenomenon.

By setting

$$\tilde{n}_2 = n_2 - iu_2 \tag{10}$$

in equation (9) we obtain

$$\cos\theta_2 = \pm \left\{ \left(1 - \frac{n_1^2 \sin^2\theta_1 (n_2^2 - u_2^2)}{(n_2^2 + u_2^2)^2}\right) - i \left(\frac{2 n_2 u_2 n_1^2 \sin^2\theta_1}{(n_2^2 + u_2^2)^2}\right) \right\}^{1/2} \tag{11}$$

or

$$\cos\theta_2 = \pm \sqrt{a - ib} = \pm \sqrt{\tilde{A}} \tag{11a}$$

Quantity b is always positive although quantity a can be positive or negative. Under these circumstances the complex vector $\tilde{A}$ can be represented by a vector lying in the third or fourth quadrant of the complex plane (see Fig. 2, vector $\tilde{A}_3$ or $\tilde{A}_4$). In both cases the square roots of $\tilde{A}$ will lie in the second or fourth quadrant.

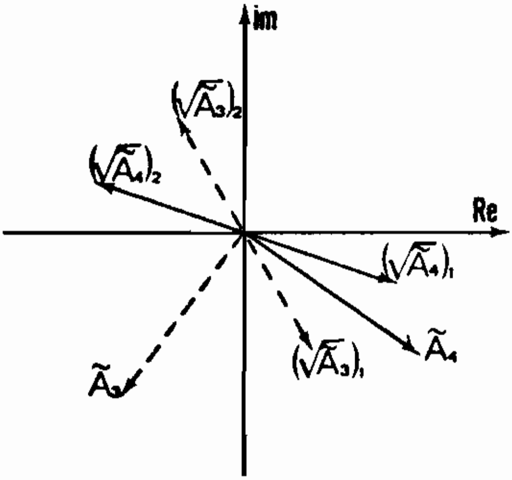

Fig. 2. Possible positions of $\cos\theta_2 = \pm\sqrt{\tilde{A}}$ *when a plane wave is propagated from a dielectric media into a conductor.*

We shall prove that only the square root of $\tilde{A}$ which is represented by a vector lying in the fourth quadrant, corresponds to the real one.

Let the interface between the two media lie in the xy plane and the plane of incidence be the zy plane, as shown in Fig. 3, where $\bar{E}$ represents the electric field (π-case) and k_1, k_r and k_t the wave vectors of the incident, reflected and transmitted light respectively.

The continuity requirements on E_{tan} across the interface at $z = 0$, lead to the following conditions;

$$k_{rx} = k_{tx} = 0 \tag{12}$$

and

$$k_{1y} = k_{ry} = k_{ty} \tag{12a}$$

Hence the components of the wave vectors will be

$$\bar{k}_1 \left(k_{1x} = 0, \quad k_{1y} = n_1 \frac{\omega}{c} \sin\theta_1, \qquad k_{1z} = \quad n_1 \frac{\omega}{c} \cos\theta_1\right)$$

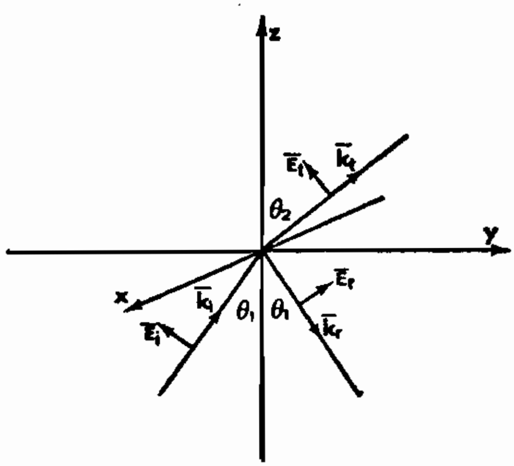

Fig. 3. Refraction and reflection of a plane wave with electric field $\bar{E}$ parallel to the plane (y, o, z) of incidence (π-case).

$$\bar{k}_r \left(k_{rx} = 0, \quad k_{ry} = n_1 \frac{\omega}{c} \sin\theta_1, \qquad k_{rz} = - n_1 \frac{\omega}{c} \cos\theta_1 \right) \qquad (13)$$

$$\bar{k}_t \left(k_{tx} = 0, \quad k_{ty} = \tilde{n}_2 \frac{\omega}{c} \sin\theta_2, \qquad k_{tz} = \quad \tilde{n}_2 \frac{\omega}{c} \cos\theta_2 \right)$$

From Eqs (13) we obtain

$$a \equiv \frac{k_{tz}}{k_{1z}} = \frac{\tilde{n}_2 \cos\theta_2}{n_1 \cos\theta_1} = \frac{(n_2 - iu_2)(c + id)}{n_1 \cos\theta_1}$$

where $(c + id)$ stands for $\cos\theta_2$. Or

$$k_{tz} = \frac{n_2 c + u_2 d}{n_1 \cos\theta_1} k_{1z} + i \frac{(n_2 d - u_2 c)}{n_1 \cos\theta_1} k_{1z} \qquad (14)$$

The full complex transmitted electric field is then given by

$$E_t(y,z,t) = (E_t)_{z=0} \exp\{i(\omega t - \bar{k}_t \cdot \bar{r})\} =$$

$$= (E_t)_{z=0} \exp\{i(\omega t - y k_{ty} - z k_{tz})\} \qquad (15)$$

where

$$k_{ty} = \tilde{n}_2 \frac{\omega}{c} \sin\theta_2 = n_1 \frac{\omega}{c} \sin\theta_1 = k_{1y} \qquad (16)$$

because of Snell's law and

$$k_{tz} = \tilde{n}_2 \frac{\omega}{c} \cos\theta_2$$

because of relation (13).
By replacing equation (14) and (16) in equation (15) one obtains.

$$E_t(y,z,t) = (E_t)_{z=0} \, exp\{i(\omega t - yk_{iy})\} \, .$$

$$\cdot \, exp\left\{ - i \left(\frac{n_2 c + u_2 d}{n_1 cos\theta_1} \right) zk_{1z} \right\} \, . \, exp\left\{ \left(\frac{n_2 d - u_2 c}{n_1 cos\theta_1} \, zk_{1z} \right\} \qquad (17)$$

The intensity or flux density in the second medium is then proportional to

$$J \sim |(E_t)_{z=0}|^2 \, exp\left\{ \frac{2(n_2 d - u_2 c) zk_{1z}}{n_1 cos\theta_1} \right\} \qquad (18)$$

The need to have a decreasing exponential dependence in the z direction for the transmitted wave leads to the conclusion

$$\frac{2(n_2 d - u_2 c) zk_{1z}}{n_1 cos\theta_1} < 0$$

or

$$n_2 d < u_2 c \qquad (19)$$

The latter inequality must hold also for all small values of u_2 or when $u_2 \rightarrow 0$. Therefore

$$n_2 d < 0 \qquad (20)$$

Since $n_2 > 0$ it follows that $d < 0$.

As $cos\theta_2$ must lie in the second or fourth quadrant, and since $d < 0$, the quantity c ought to be positive $(c > 0)$. Finally $cos\theta_2 = c - id$ with c and d real positive numbers which determine the position of $cos\theta_2$ in the fourth quadrant. (see Fig. 1, point (3)).

c. Finally let us consider the propagation of a plane wave from a dielectric (n_1) into a conductor $(\tilde{n}_2)$, the surface of contact between them being the plane $z = 0$, and from this conductor into a second conductor $(\tilde{n}_3)$, the surface of contact between these two conductors being the plane $z = C$.

By applying the law of refraction we have

$$\tilde{n}_1 sin\theta_1 = \tilde{n}_2 sin\theta_2 = \tilde{n}_3 sin\theta_3 \tag{21}$$

or

$$cos\theta_3 = \pm \left\{ 1 - \left(\frac{n_1}{n_3} \, sin\theta_1 \right)^2 \right\}^{1/2} \tag{22}$$

Taking the z components of the wave vectors $\overline{k}_1$, $\overline{k}_2$ and $\overline{k}_3$ we have

$$a_{32} \equiv \frac{k_{3z}}{k_{2z}} = \frac{\tilde{n}_3 cos\theta_3}{\tilde{n}_2 cos\theta_2} \tag{23}$$

Similarly

$$a_{21} \equiv \frac{k_{2z}}{k_{1z}} = \frac{\tilde{n}_2 cos\theta_2}{n_1 cos\theta_1} \tag{24}$$

or

$$k_{3z} = a_{32} \cdot a_{21} \cdot k_{1z} = \frac{\tilde{n}_3 cos\theta_3}{\tilde{n}_2 cos\theta_2} \cdot \frac{\tilde{n}_2 cos\theta_2}{n_1 cos\theta_1} \cdot k_{1z} = \frac{\tilde{n}_3 cos\theta_3}{n_1 cos\theta_1} \cdot k_{1z}$$

So we ended up again with the investigation of the propagation of a plane wave from a dielectric into a conductor which leads to the conclusion that $cos\theta_3$ is a quantity of the form

$$cos\theta_3 = c_3 - id_3$$

where c_3 and d_3 are positive and real numbers.

Consequently the values of $cos\theta_3$ refer to the fourth quadrant of the complex plane.

CONCLUSIONS

In all cases we come to the conclusion that the value of the cosine of the refracted angle must lie in the fourth quadrant of the complex plane.

The limit values of $cos\theta_2$ correspond to the propagation of light from an optically less dense medium to a denser one and from an optically dense medium to a less dense one, under the condition that the angle of incidence is larger than the critical angle. In these cases the limit values are represented by points lying in the positive part of the real axis and in the negative part of the imaginary axis respectively. In all other cases $cos\theta_2$ is represented by points lying in the fourth quadrant area.

RAFERENCES

1. Born, M. and Wolf, E.: Principles of Optics (4th edition 1970), p. 38 (Pergamon Press).
2. Klein, M. V.: Optics, p. 576 (1970), (J. Wiley & Sons Inc).

ΠΕΡΙΛΗΨΙΣ

ΠΕΡΙ ΤΗΣ ΠΑΡΑΣΤΑΣΕΩΣ ΤΟΥ ΣΥΝΗΜΙΤΟΝΟΥ ΤΗΣ ΓΩΝΙΑΣ ΔΙΑΘΛΑΣΕΩΣ ($cos\theta_2$) ΕΙΣ ΤΟ ΜΙΓΑΔΙΚΟΝ ΕΠΙΠΕΔΟΝ

Ὑπὸ

Ι. ΣΑΡΑΚΙΝΟΥ καὶ Ι. ΣΠΥΡΙΔΕΛΗ

('Εργαστήριον Α' ἕδρας Φυσικῆς, Πανεπιστημίου Θεσσαλονίκης)

Διὰ τὴν μελέτην φαινομένων συμβολῆς ἐντὸς ὑμενίων ἐφαρμόζεται ἡ γνωστὴ σχέσις

$$\delta = \frac{2\pi n_2 d cos\theta_2}{\lambda}$$

ἡ παρέχουσα τὴν διαφορὰν φάσεως μεταξὺ τῆς ἀνακλωμένης καὶ διαθλωμένης ἀκτινοβολίας. Εἰς τὴν ἐν λόγω σχέσιν, τὸ μέγεθος ($cos\theta_2$), ἔνθα θ_2 ἡ γωνία διαθλάσεως, εἶναι ὡς ἐπὶ τὸ πλεῖστον φανταστικὸν μέγεθος τῆς μορφῆς $cos\theta_2 = c - id$.

Εἰς πλεῖστα προβλήματα ἰδίως εἰς προγράμματα ὑπολογιστῶν ἀναφερόμενα εἰς προβλήματα ἐλλειψομετρίας, εἶναι ἀπαραίτητον νὰ γνωρίζωμεν ἐὰν τὰ μεγέθη c, d εἶναι θετικὰ ἀρνητικὰ ἢ μηδέν.

Διακρίνομεν τὰς κάτωθι περιπτώσεις:

(α) Κατὰ τὴν διάδοσιν τῆς ἀκτινοβολίας ἔκ τινος διηλεκτρικοῦ δ.δ. n_1 εἰς ἕτερον δ.δ. n_2, διὰ τὰ ὁποῖα ἰσχύει ἡ συνθήκη $n_1 < n_2$ τότε

$$cos\theta_2 = c - i0$$

ἔνθα c θετικὸς πραγματικὸς ἀριθμός.

'Εὰν ὅμως $n_1 > n_2$ καὶ ἡ γωνία προσπτώσεως θ_1 τῆς ἀκτινοβολίας εἶναι μεγαλυτέρα τῆς ὁρικῆς γωνίας $\theta_C (\theta_1 > \theta_C)$ τότε,

$$cos\theta_2 = 0 - id$$

ἔνθα d θετικὸς πραγματικὸς ἀριθμός.

(β) Κατὰ τὴν διάδοσιν τῆς ἀκτινοβολίας ἔκ τινος διηλεκτρικοῦ μέσου δ.δ. n_1 εἰς ἓν ἀγώγιμον τοιοῦτον δ.δ. $\tilde{n}_2$ (ἴδε σχῆμα 3) τότε ἀποδεικνύεται ὅτι,

$$\cos\theta_2 = c - id$$

ἔνθα c καὶ d πραγματικοὶ θετικοὶ ἀριθμοί.

(γ) Τέλος κατὰ τὴν διάδοσιν τῆς ἀκτινοβολίας ἔκ τινος ἀγωγίμου μέσου δ.δ. $\bar{n}_1$ εἰς ἄλλο δ.δ. $\bar{n}_2$, θὰ ἰσχύη καὶ πάλιν ὅτι,

$$\cos\theta_2 = c - id$$

ἔνθα c καὶ d θετικοὶ καὶ πραγματικοὶ ἀριθμοί.

Συνεπῶς εἰς ὅλας τὰς περιπτώσεις διαδόσεως μιᾶς ἀκτινοβολίας ἔκ τινος μέσου εἰς ἕτερον τοιοῦτον τὸ συνημίτονον τῆς γωνίας διαθλάσεως θ_2 $(\cos\theta_2)$ εἶναι δυνατὸν νὰ παρασταθῇ εἰς τὸ μιγαδικὸν ἐπίπεδον ὑφ' ἑνὸς διανύσματος κειμένου εἰς τὸ τέταρτον τεταρτημόριον.

Ἡ μία ἀκραία θέσις τοῦ διανύσματος συμπίπτει μετὰ τοῦ θετικοῦ πραγματικοῦ ἄξονος τοῦ μιγαδικοῦ ἐπιπέδου καὶ ἀναφέρεται εἰς τὴν περίπτωσιν διαδόσεως τῆς ἀκτινοβολίας ἔκ τινος διηλεκτρικοῦ δ.δ. n_1 εἰς ἕτερον μεγαλυτέρου δ.δ. n_2. Ἡ δευτέρα ἀκραία θέσις τοῦ διανύσματος συμπίπτει μετὰ τοῦ ἀρνητικοῦ φανταστικοῦ ἄξονος καὶ ἀναφέρεται εἰς τὴν περίπτωσιν διαδόσεως τῆς ἀκτινοβολίας ἔκ τινος διηλεκτρικοῦ δ.δ. n_1 εἰς ἕτερον μικροτέρου δ.δ. n_2, ἡ δὲ γωνία προσπτώσεως εἶναι μεγαλυτέρα τῆς ὁριχῆς.